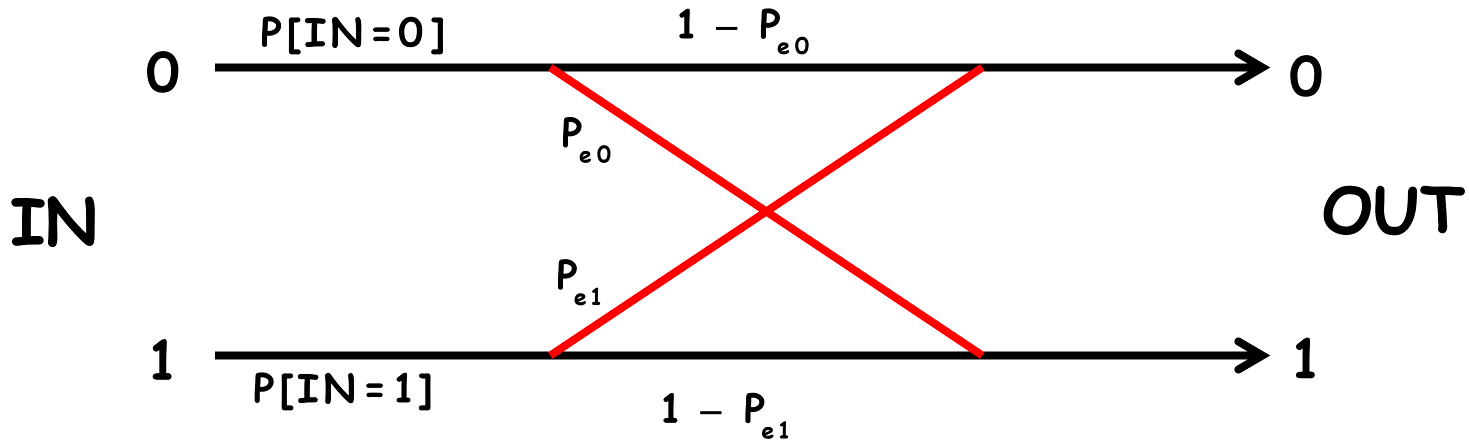
ELEC1200: A System View of Communications: from Signals to Packets
Lecture 9

- **Review:**
  - Analyzing Bit Errors in the Binary Channel Model
- **Inside the Binary Channel**
  - Average Power in Signals
  - Modeling random noise
  - Calculating the probability of error conditioned on the input
- **Calculating bit error rate**
  - Effect of changing threshold
  - Effect of changing signal
  - Effect of changing noise
- **Signal to Noise Ratio**

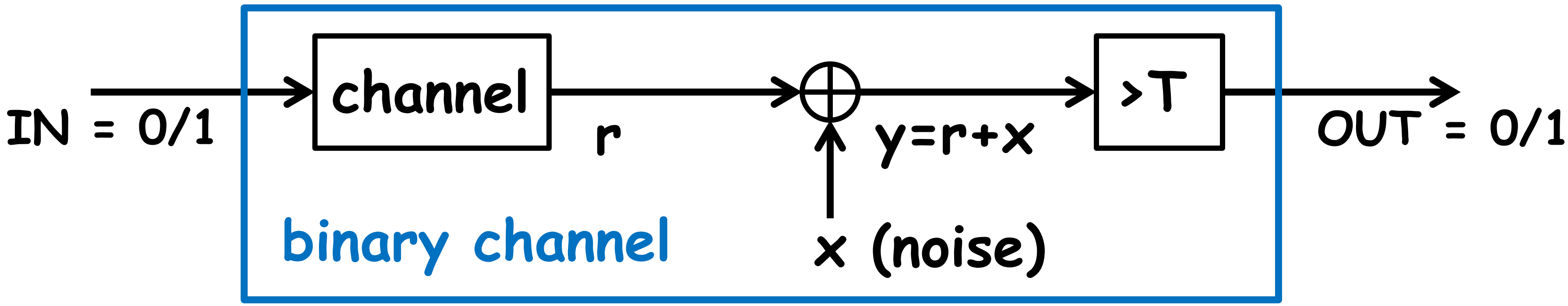
Binary Channel Model



- Binary: input/output are either 0/1.
- Usually, the input and output are the same, but occasionally (with probability P_{e0} or P_{e1}), a bit is flipped.
- The BER can be calculated by

$$BER = P_{e0} \cdot P[IN=0] + P_{e1} \cdot P[IN=1]$$

Inside the Binary Channel



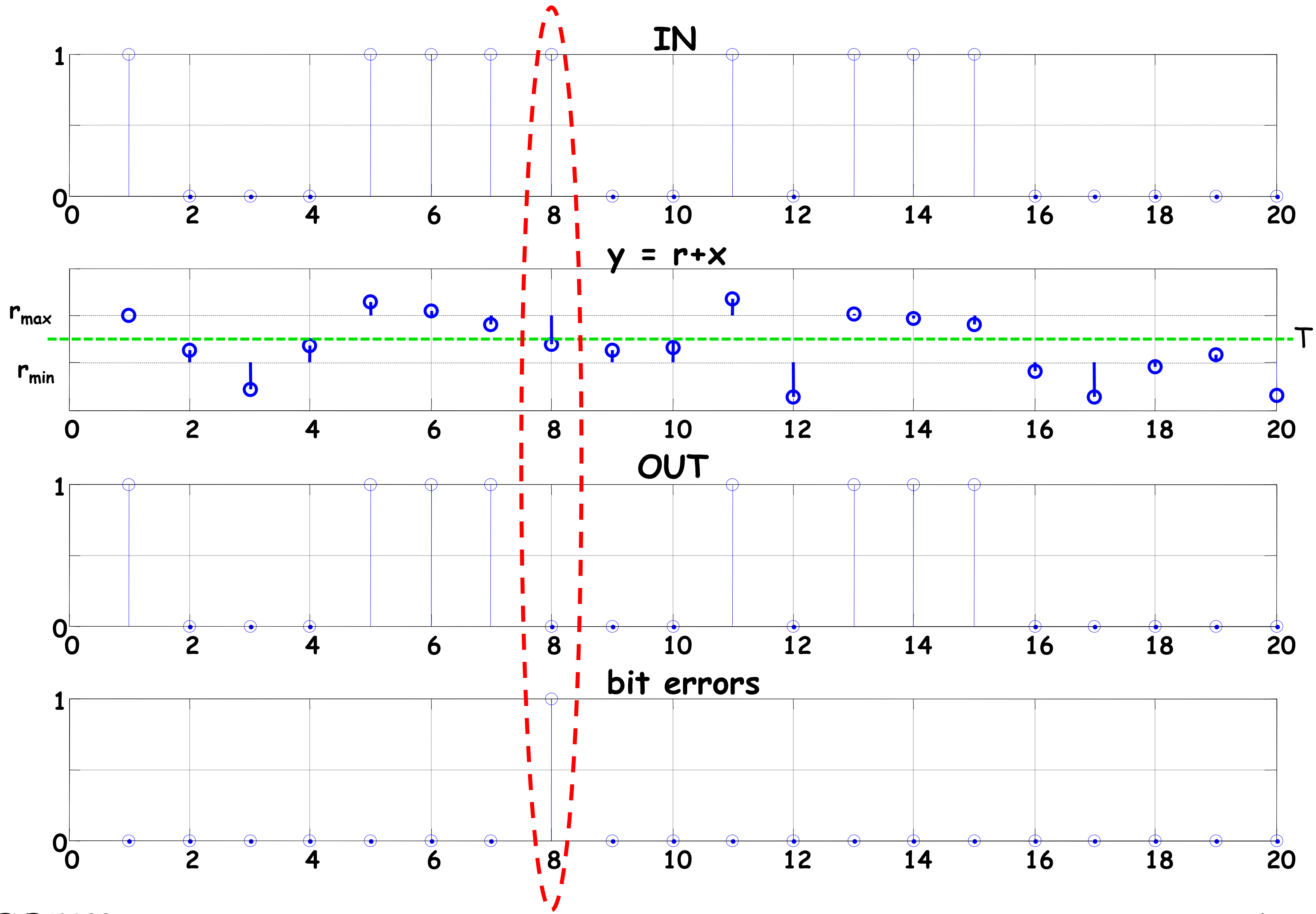
- Under our simplifying assumptions, we can consider one bit at a time.
- The channel adds an offset r_{min} and scaling by $r_{max}-r_{min}$

$$r = \begin{cases} r_{min} & \text{if } IN = 0 \\ r_{max} & \text{if } IN = 1 \end{cases}$$

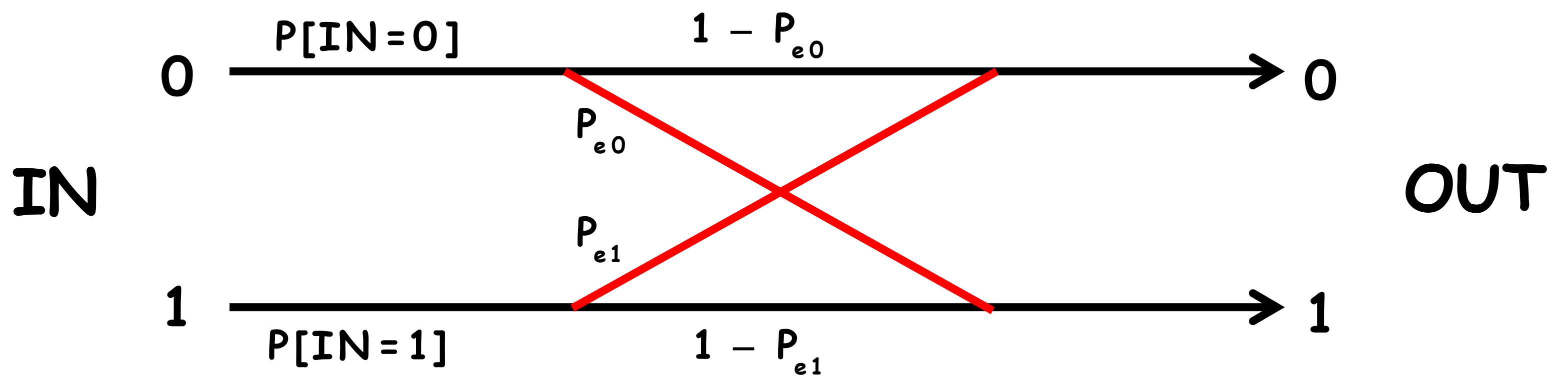
- The noise is additive: $y = r + x$
- The output is obtained by thresholding y:

$$out = \begin{cases} 0 & \text{if } y < T \\ 1 & \text{if } y \geq T \end{cases} \qquad T = \text{threshold}$$

Noise leads to bit errors



Parameters that determine BER



- In order to predict the BER, we need to know
  - P[IN=0] (we can find P[IN=1] = 1 - P[IN=0])
  - P_{e0}
  - P_{e1}
- Usually, the transmitter determines P[IN=0]
  - e.g. P[IN=0] = P[IN=1] = 0.5
- P_{e0} and P_{e1} depend on
  - the transmit levels (r_{min},r_{max})
  - the power in the noise

This lecture examines this dependency

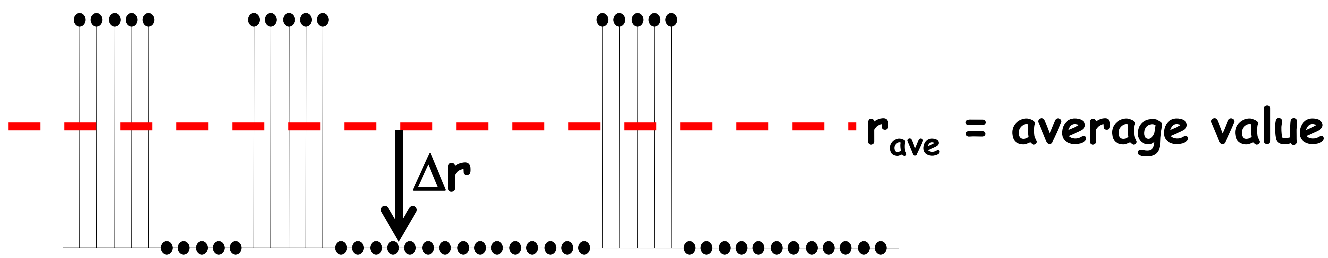
Intuition

- The probabilities of error, P_{e0} and P_{e1}, should be small if
  - The transmit levels are widely separated, i.e.
    - r_{max} - r_{min} is large
    - The power used to transmit the signal is large
  - The noise is small, i.e.
    - The values of the noise do not vary over a wide range
    - The power of the noise is small
- In this case, the BER will also be small
  - Remember that the BER lies between P_{e0} and P_{e1}

- Implicit in these definitions is the idea of the power in signals.

Average Power in Signals

- For communication, we usually have signals that vary around an average value



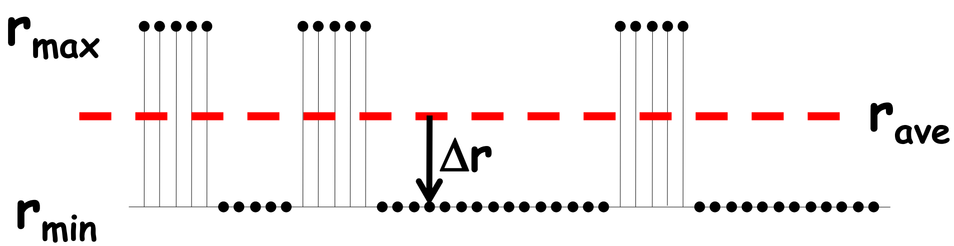
- For communication, we are interested in how much the signals differ from their average:

$$\Delta r = r - r_{ave}$$

- Since this changes over time and can be both positive and negative, we average the squared value over a number of samples

$$P = \frac{1}{N}\sum_{n=1}^{N}\left(\Delta r(n)\right)^2$$

Average Power for Bit Signals



$$P_{signal} = \frac{1}{N}\sum_{n=1}^{N}\left(\Delta r(n)\right)^2$$

- If 0 and 1's are equally likely,

$$r_{ave} = \tfrac{1}{2}r_{min} + \tfrac{1}{2}r_{max}$$

- If IN = 0,

$$\begin{aligned}\Delta r &= r_{min} - r_{ave} \\ &= r_{min} - (\tfrac{1}{2}r_{min} + \tfrac{1}{2}r_{max}) = \tfrac{1}{2}(r_{min} - r_{max})\end{aligned}$$

- If IN = 1,

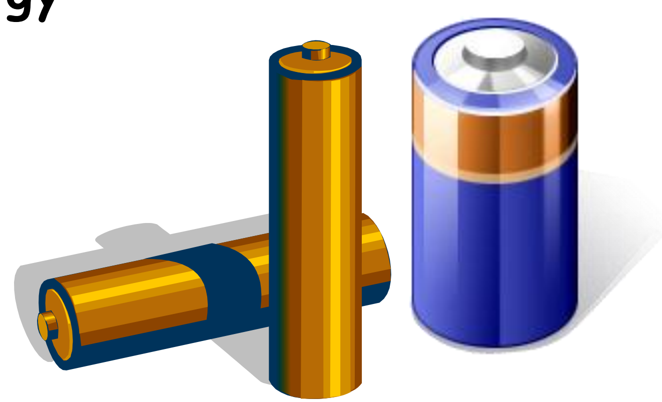
$$\Delta r = r_{max} - (\tfrac{1}{2}r_{min} + \tfrac{1}{2}r_{max}) = \tfrac{1}{2}(r_{max} - r_{min})$$

- The average power is

$$P_{signal} = \frac{1}{2}\left[\tfrac{1}{2}(r_{max} - r_{min})\right]^2 + \frac{1}{2}\left[\tfrac{1}{2}(r_{max} - r_{min})\right]^2 = \frac{\left(r_{max} - r_{min}\right)^2}{4}$$

Power Consumption

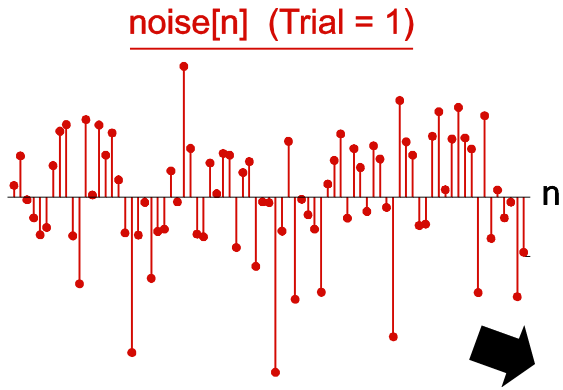
- Power consumption directly affects battery life.
- Power is energy used per unit time
- Batteries contain a fixed amount of energy. The higher the power consumption of the device they are powering, the faster this energy is used up.
- Calculating the amount of energy in a battery
  - Batteries are typically rated at fixed voltage in volts (V) and a charge capacity in milliamp-hours (mAh)
  - Multiplying these together gives total energy in milliwatt-hours (mWh)
  - For example, a mobile phone battery rated at 3.7V and 1000mAh contains 3700mWh of energy
- Typical power consumption of
  - A microwave oven 1000W
  - A desktop computer 120W
  - A notebook computer 40W
  - A human brain 10W
  - A mobile phone 1W

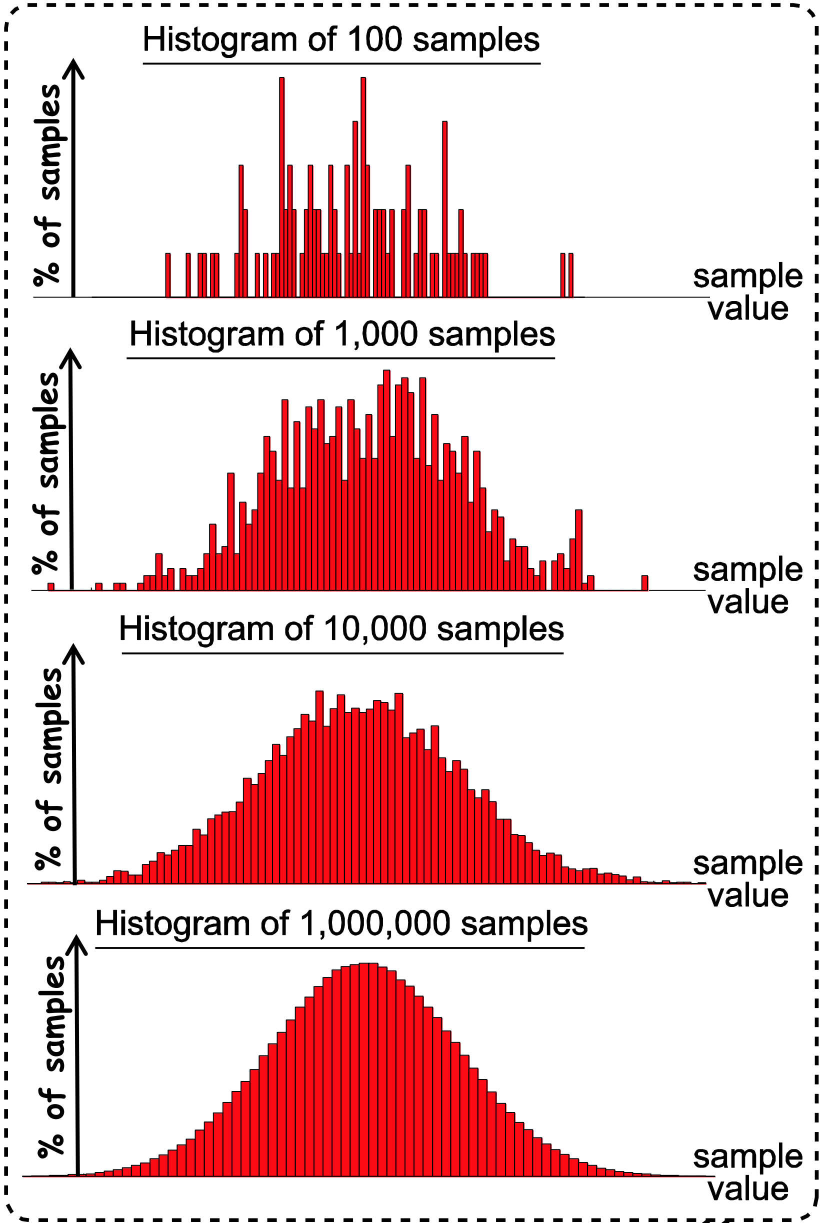


Noise levels in Mobile Phones

- It determines the minimum signal that can be received by radios and receivers
- What is the typical noise power present at the input of a mobile phone?
- Extremely, extremely small 10^{-15}W
  - 1 Watt = Unit of Power
  - Lifting an apple (~100g) up by 1 m in 1s requires approximately 1W
- When your received signal falls to about this level your phone will lose its connection
- The exact level is determined by
  - Quality of circuits and components
  - Symbol (bit) rate

Statistics of the Noise

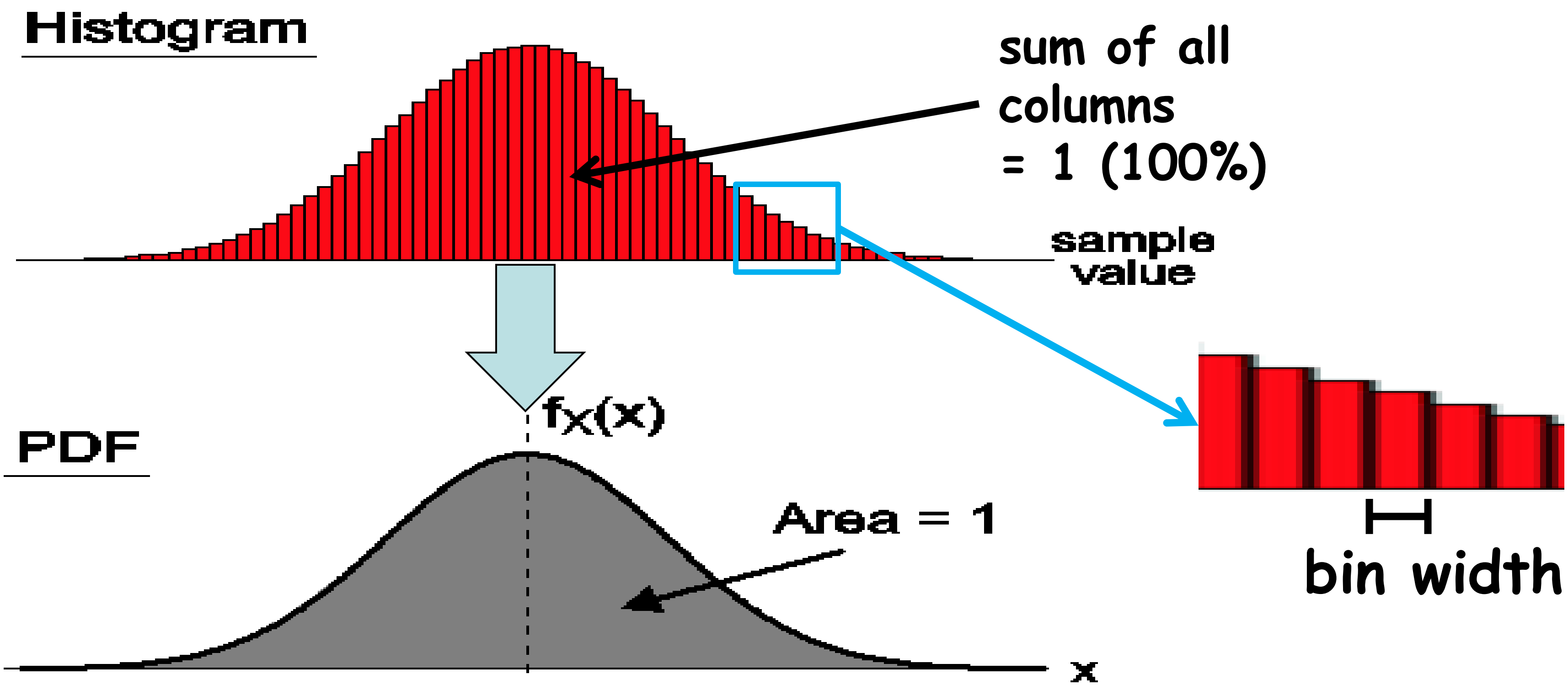




- Even though the value of the noise is quite random, the statistics of a large number of samples are quite stable.
- Create histograms of samples
  - Height at each sample value is the percentage of samples near that sample value.
- As we collect more samples, the histogram gets smoother and smoother.

Probability Density Function

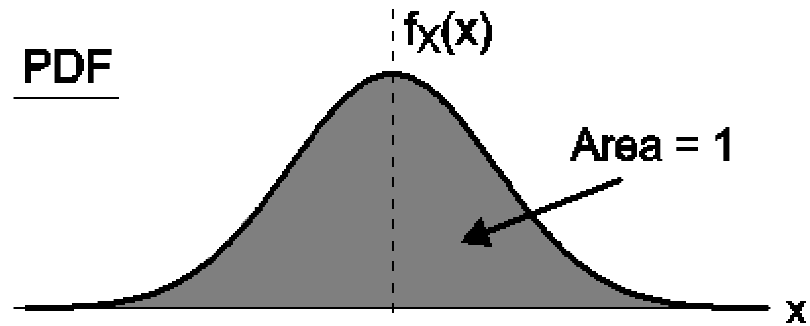
- The histogram is still not smooth since we count the samples in bins of finite width.
- As the bins get smaller and smaller, the curve gets smoother and smoother, and approaches a function known as the probability density function (pdf)



Gaussian Density Function

- The probability density function of many naturally occurring random quantities, such as noise or measurement errors, tends to have a hill or bell-like shape, known as a Gaussian distribution.
- This very important result is due to a mathematical result known as the Central-Limit Theorem.
- This random variable is so common that it is also called the "normal" random variable.
- Applications:
  - Noise voltages that corrupt transmission signals.
  - Position of a particle undergoing Brownian motion
  - Voltage across a resistor

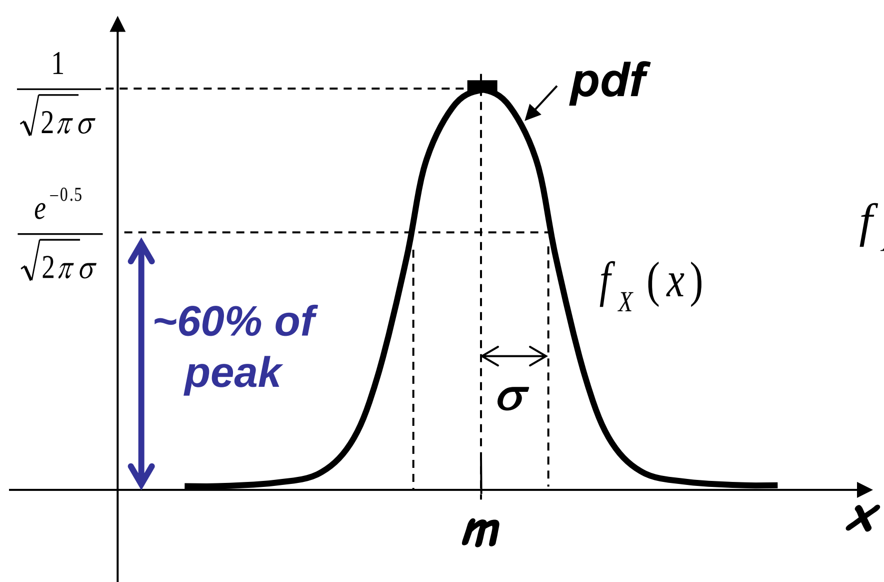
Gaussian PDF



The wider the PDF, the larger the noise.

Parameters controlling the shape

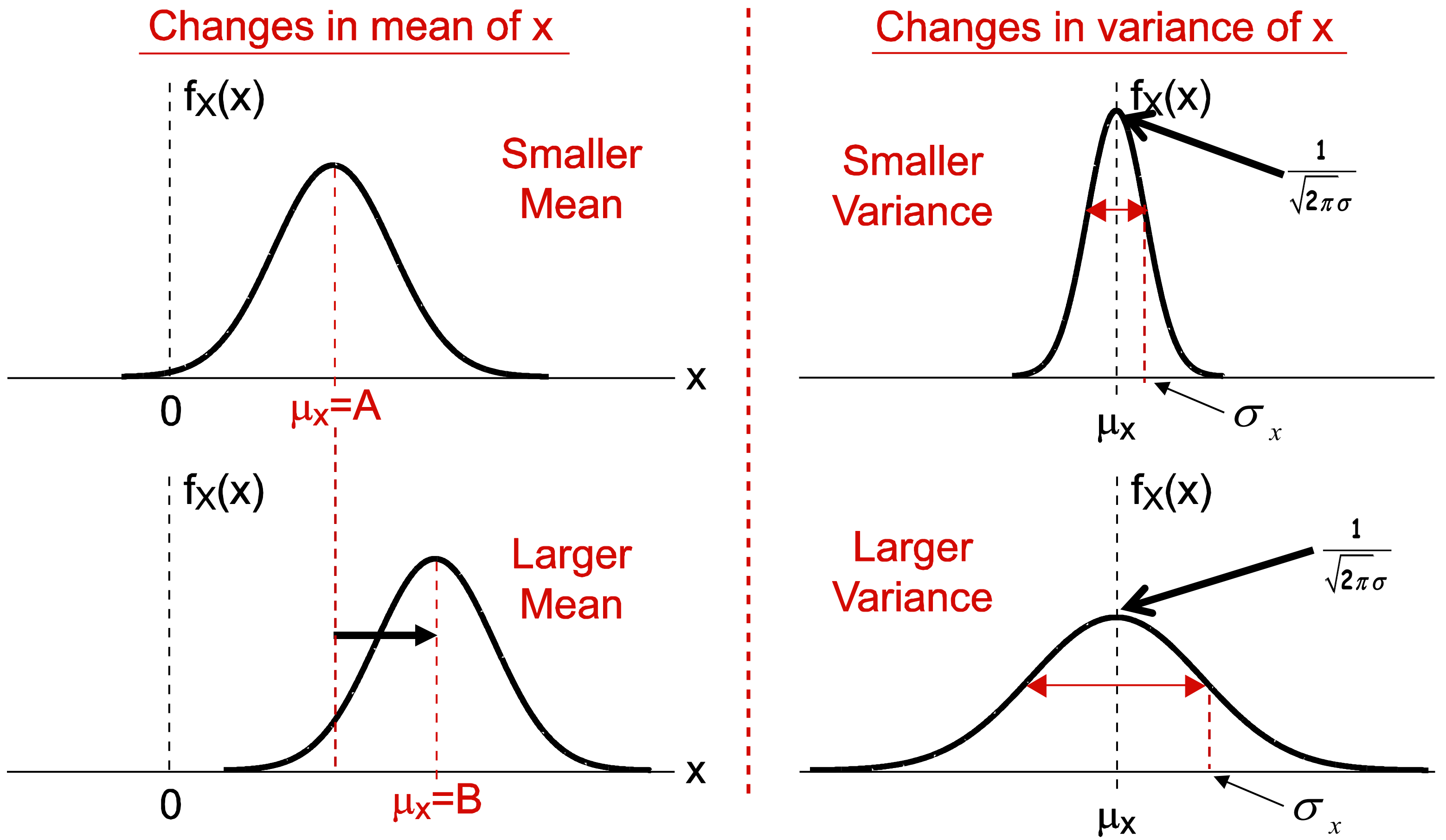
- The mean (m) of a Gaussian random variable is
  - Its average value over many samples
  - The center location of the pdf
- The standard deviation (σ) is
  - An indication of how "spread out" the samples are
  - A measure of the width of the pdf
- The variance (σ^2) is
  - The square of the standard deviation
  - The average power over many samples



$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}$$

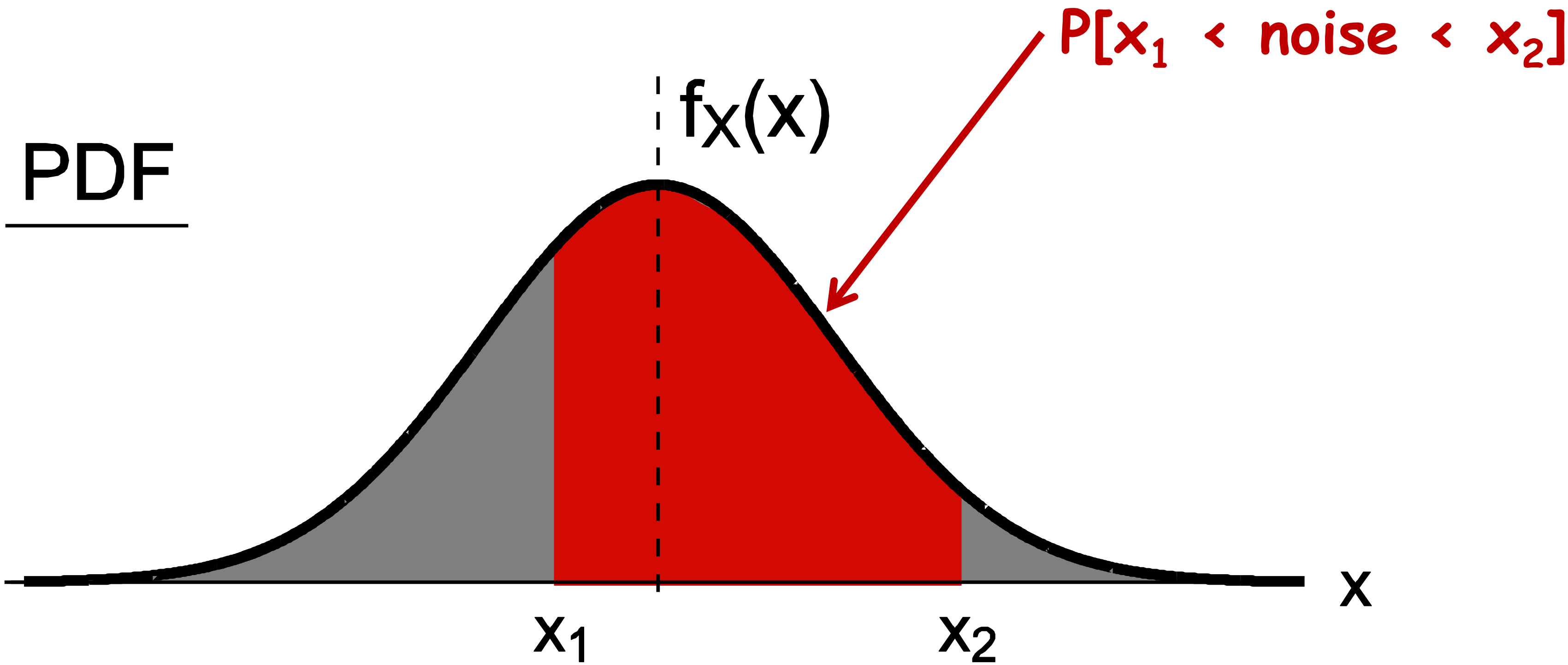
Changing the Mean and Variance

- Changes in mean shift the *center of mass* of PDF
- Changes in variance narrow or broaden the PDF
  - Note that area of PDF must always remain equal to one

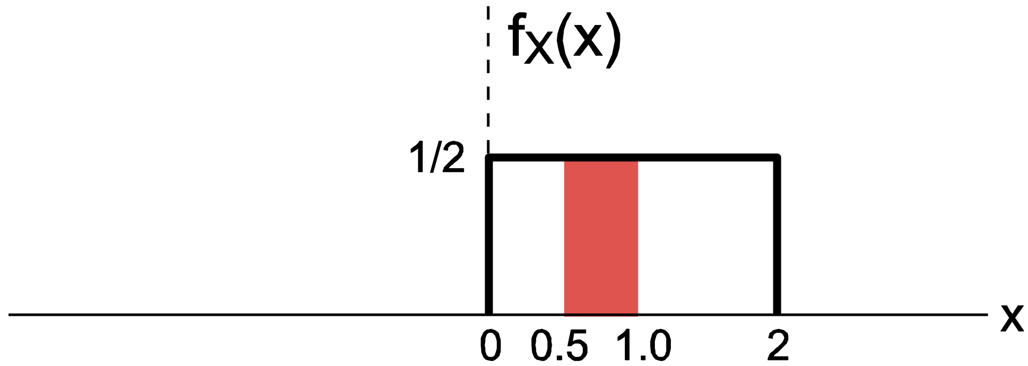


Calculating Probability by Integrating

- The probability that the noise x is in (x_1, x_2) is the area under the probability density function between x_1 and x_2



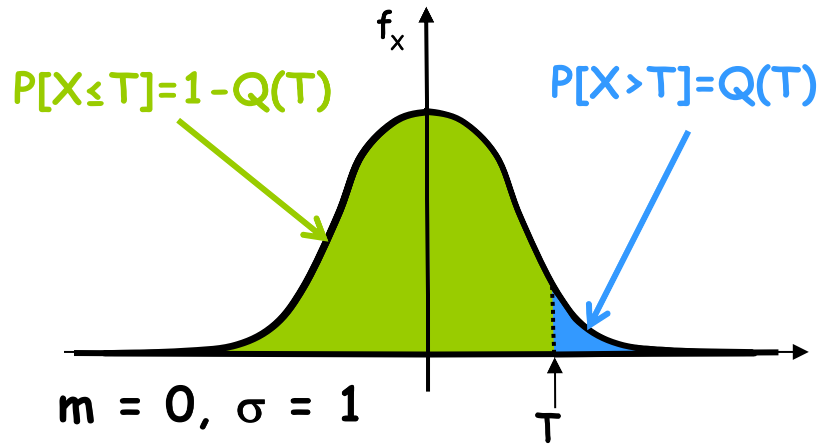
Example Probability Calculation



- **Verify that overall area is 1:**
  - **Since the curve defines a rectangle, the area is base × height:** $2 \times \frac{1}{2} = 1$
- **Probability that x is between 0.5 and 1.0:**
  - **The area of the shaded region is** $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
  - **Thus,** $P\left[0.5 < x < 1.0\right] = \frac{1}{4}$

The Q-function

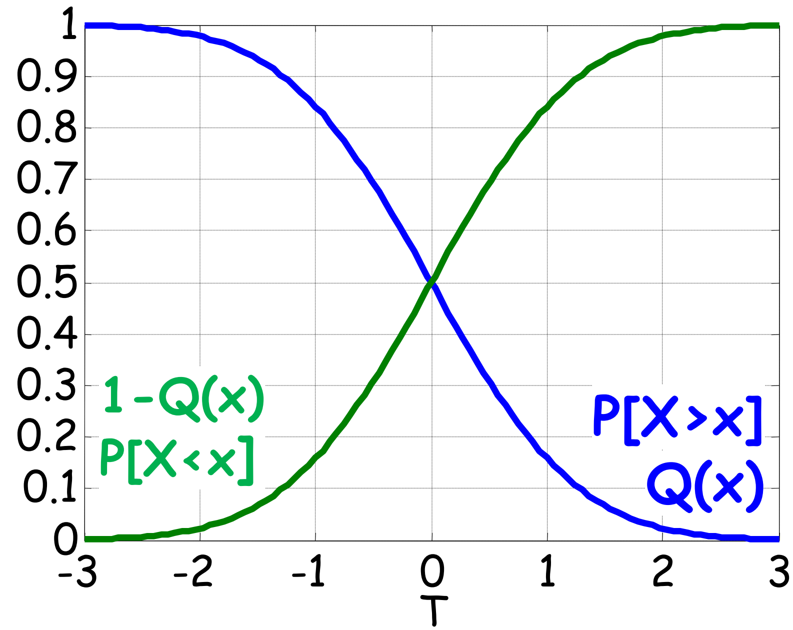
- The Q-function gives the probability that a Gaussian random variable X with $m = 0$ and $\sigma = 1$ is greater than a particular value T



- Properties

  $P[X \le T] = 1 - Q(T)$

  $Q(0) = \frac{1}{2}$



- There is no closed-form expression for Q(T). Its value must be found from tables or the MATLAB function

```
qfunc(T)
```

Linear Transformations of Gaussian

- Suppose X is Gaussian with m=0 and σ=1.
- If Y=aX+b, then

  Y is Gaussian

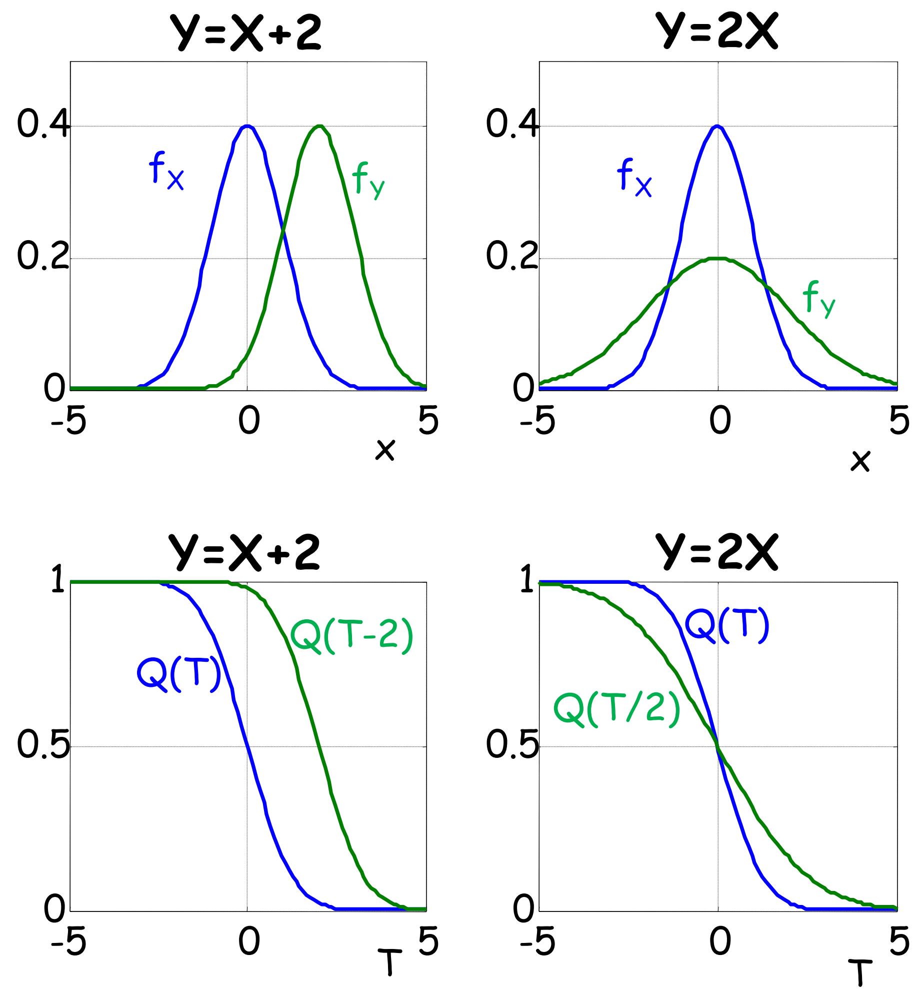
  with m=b and σ=a!

- We can compute probabilities for X using the Q function, e.g.

$$P[X > T] = Q(T)$$

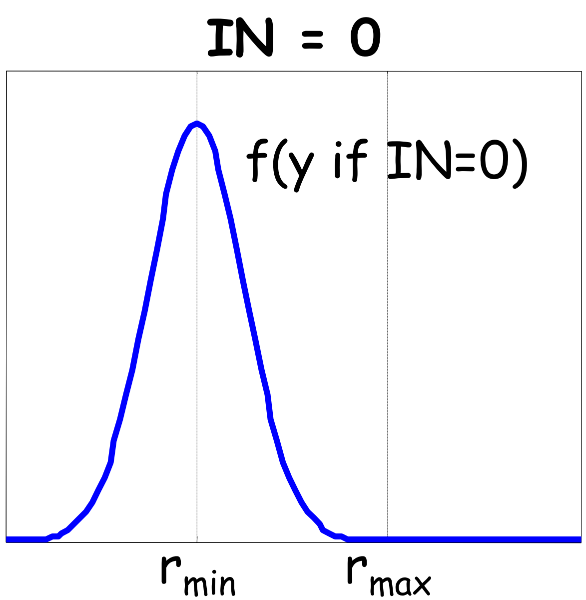
- We can also compute probabilities for Y using the Q function, since

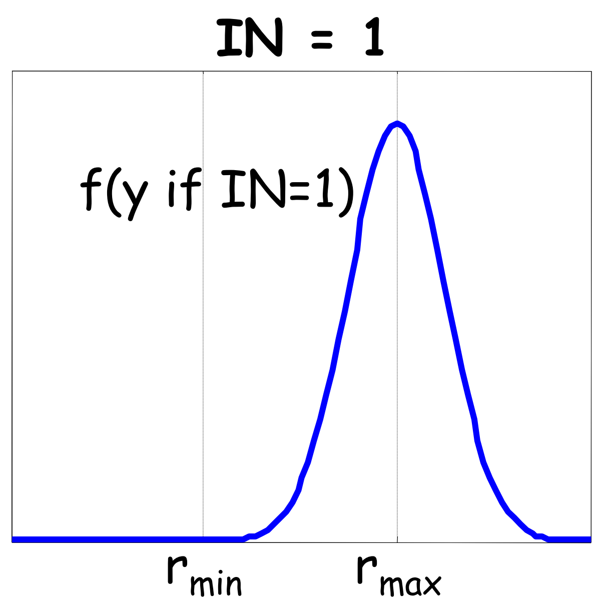
$$P[Y > T] = Q\left(\frac{T - m}{\sigma}\right)$$



PDF of Received Signal + Noise

- Assume the noise x is Gaussian with zero mean and variance (power) σ^2.
- The PDF of $y = r+x$ depends upon whether IN = 0 or 1
  - If IN=0, $y = r_{min} + x$
    - y is Gaussian with mean r_{min} and variance σ^2
  - If IN=1, $y = r_{max} + x$
    - y is Gaussian with mean r_{max} and variance σ^2





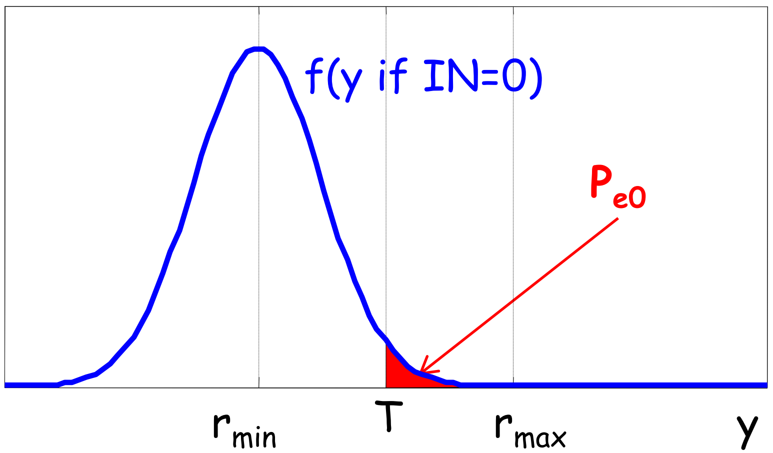
P_{e0} (Probability of Error if IN=0)

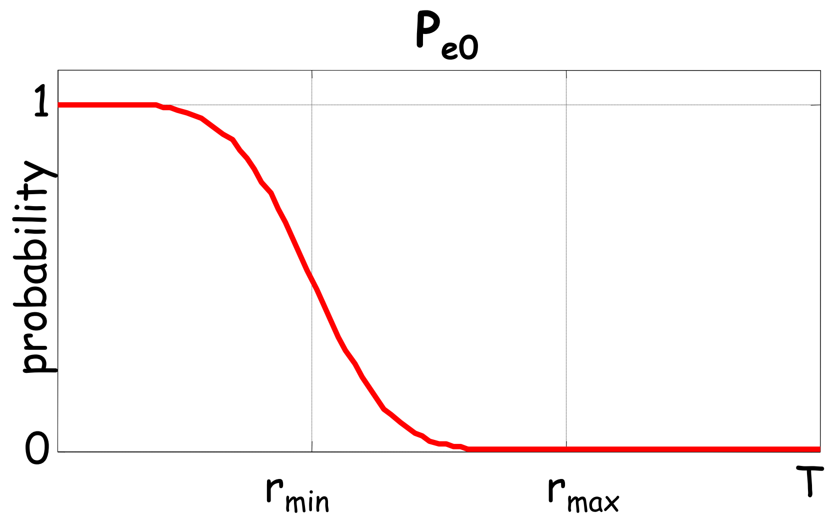
- If IN = 0,
  - $y = r_{min} + x$
  - y is a Gaussian with mean r_{min} and variance σ^2

- There is an error if
  - OUT = 1
  - The noise pushes y above T

$$P_{e0} = P[y > T \text{ if } IN = 0]$$
$$= Q\left(\frac{T - r_{min}}{\sigma}\right)$$

- The probability of error decreases as T increases.





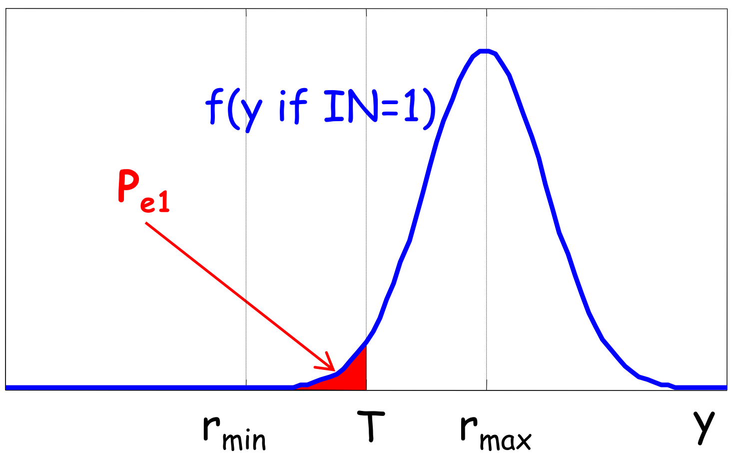
P_{e1} (Probability of Error if IN=1)

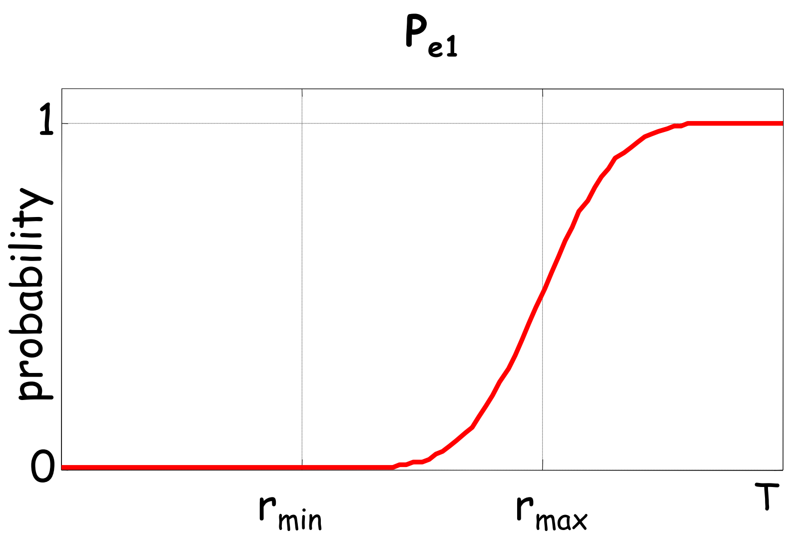
- If IN = 1,
  - $y = r_{max} + x$
  - y is a Gaussian with mean r_{max} and variance σ^2

- There is an error if
  - OUT = 0
  - The noise pushes y <u>below</u> T

$$P_{e1} = P[y < T \text{ if } IN = 1]$$
$$= 1 - Q\left(\frac{T - r_{max}}{\sigma}\right)$$

- The probability of error <u>increases</u> with T.
- Thus, choosing T is a tradeoff between minimizing P_{e0} and P_{e1}.

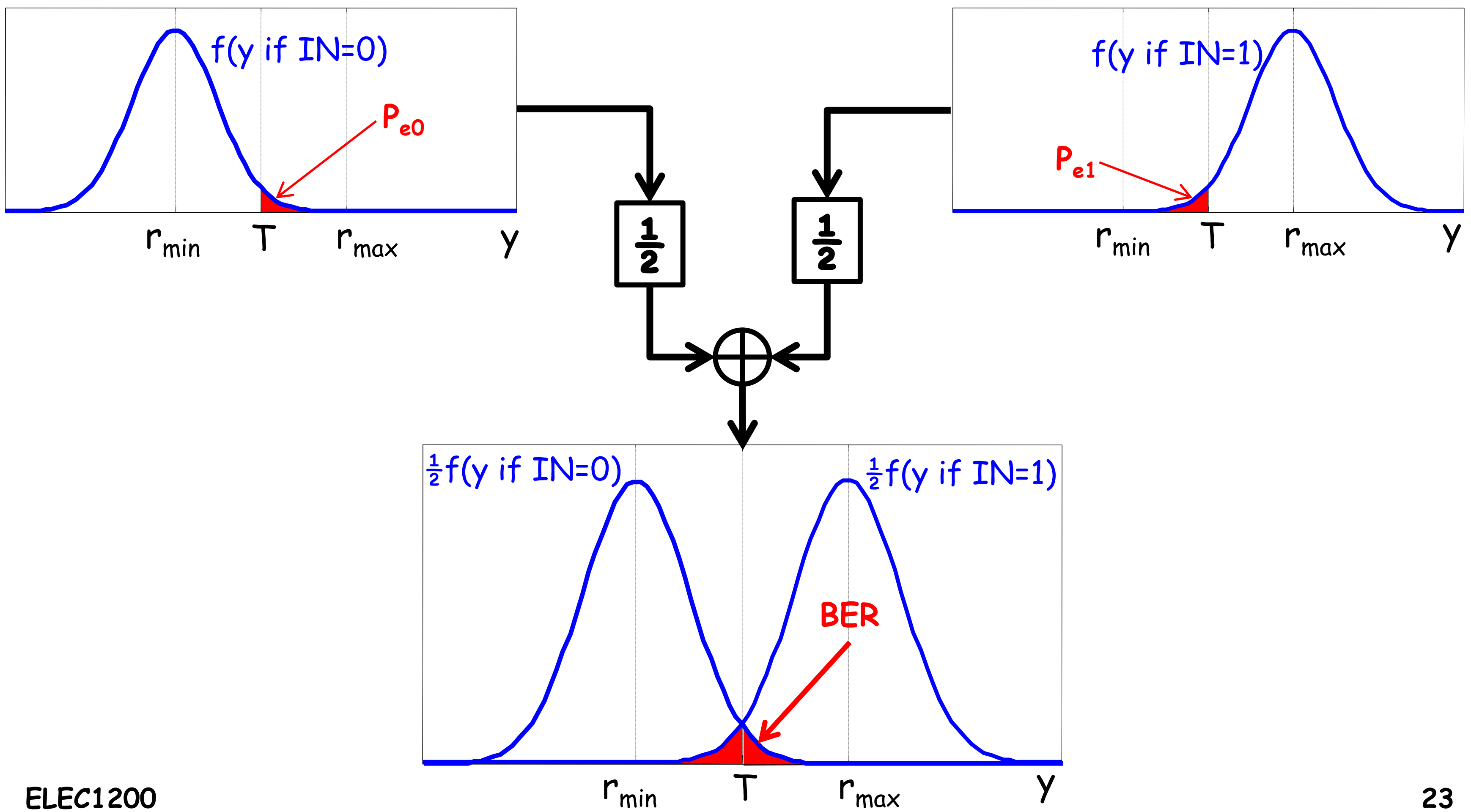




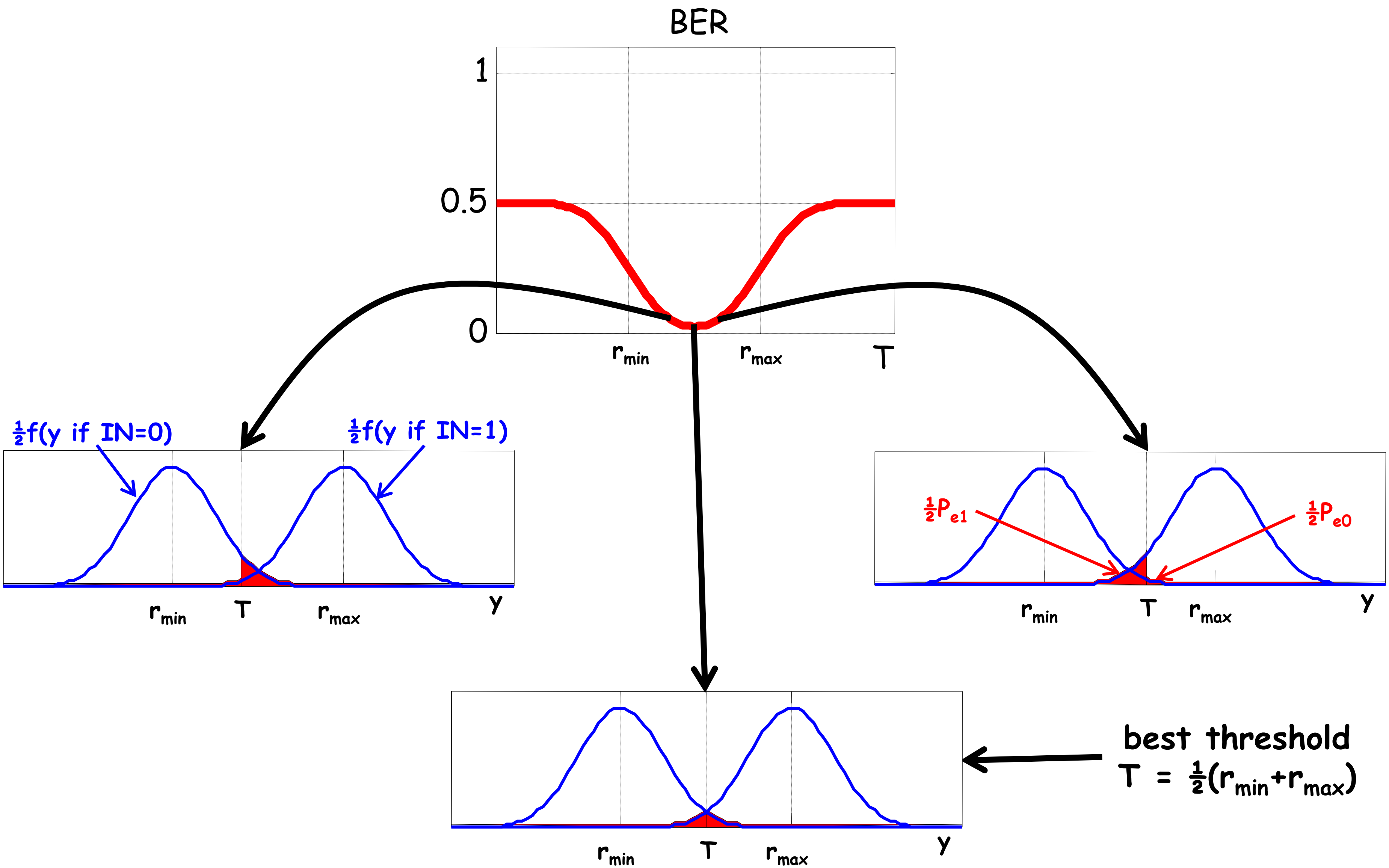
Predicting BER

If 0 and 1 input bits are equally likely,

$$\mathrm{BER} = \tfrac{1}{2} P_{e0} + \tfrac{1}{2} P_{e1}$$

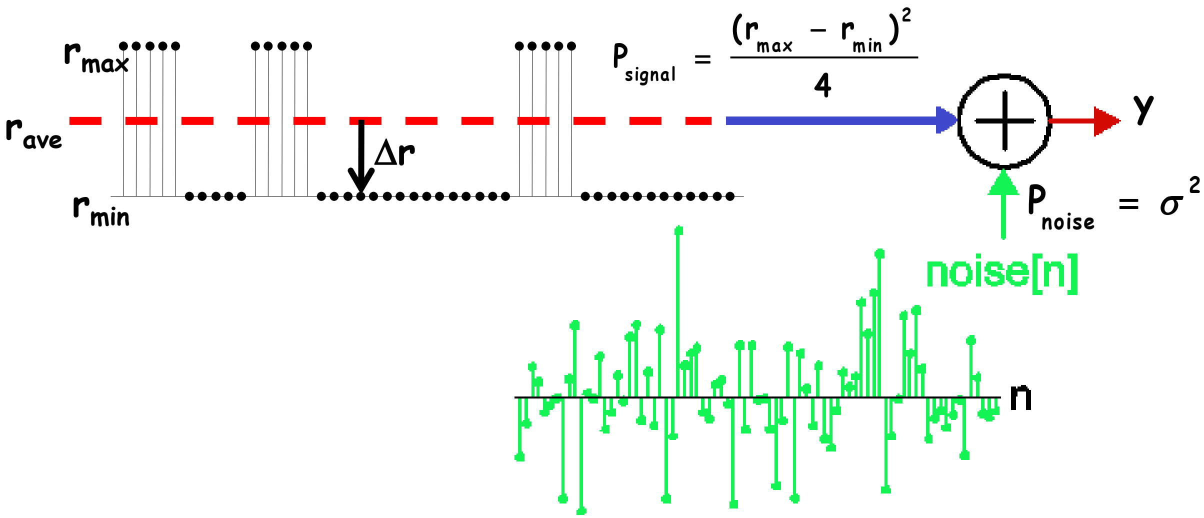


BER under changing threshold



Signal-to-Noise Ratio

- It is not the absolute signal or noise power that is important, but rather the Signal-to-Noise Ratio (SNR).

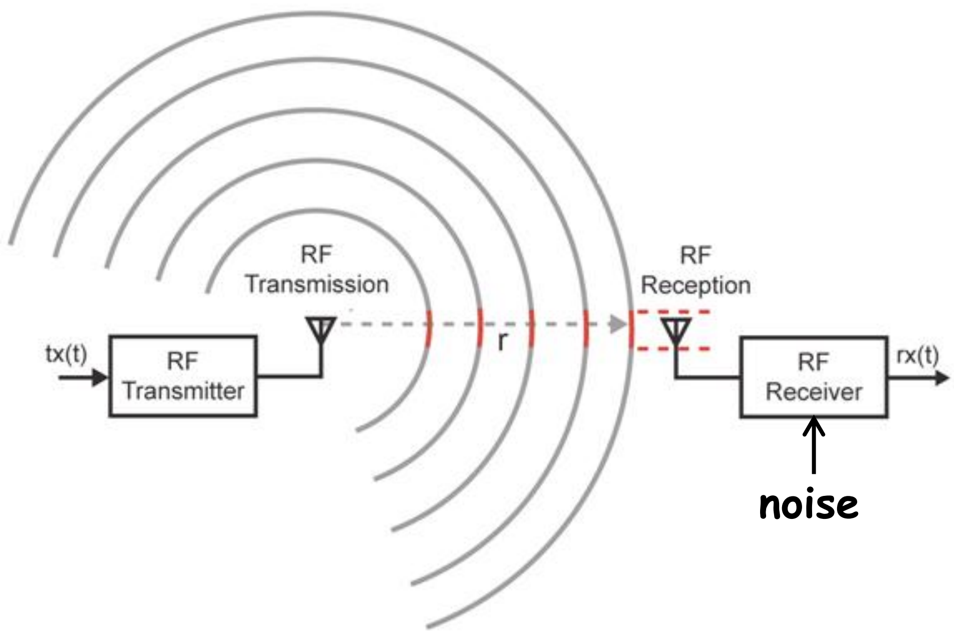


$$SNR = \frac{P_{signal}}{P_{noise}} = \frac{(r_{max} - r_{min})^2}{4\sigma^2}$$

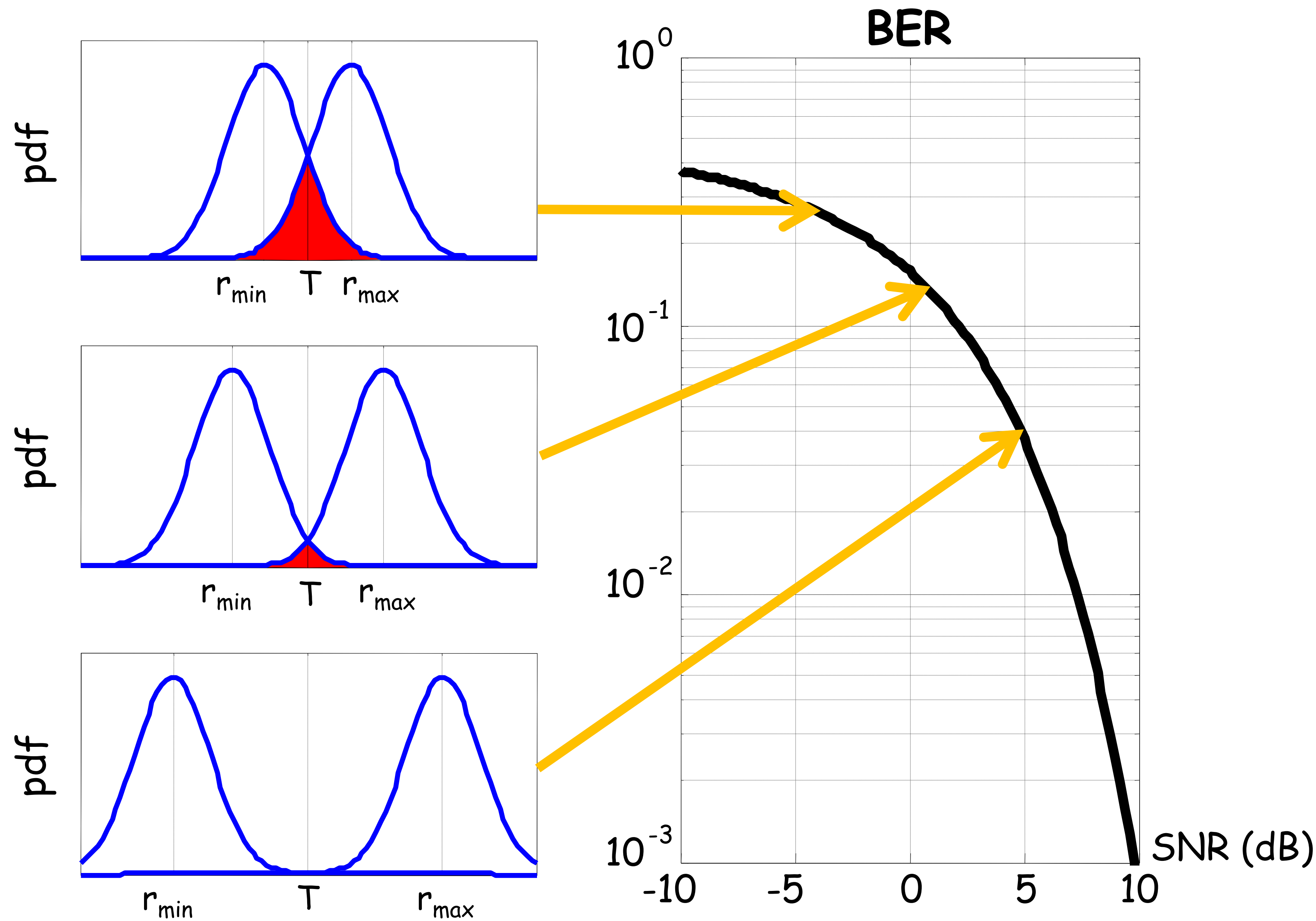
- SNR is often measured in decibels (dB): $SNR = 10 \log_{10} \frac{P_{signal}}{P_{noise}}$ (dB)
  - SNR = 0dB means signal and noise power are equal
  - SNR = 10dB means signal power is 10 times noise power
  - SNR = 20dB means signal power is 100 times noise power

Factors affecting SNR

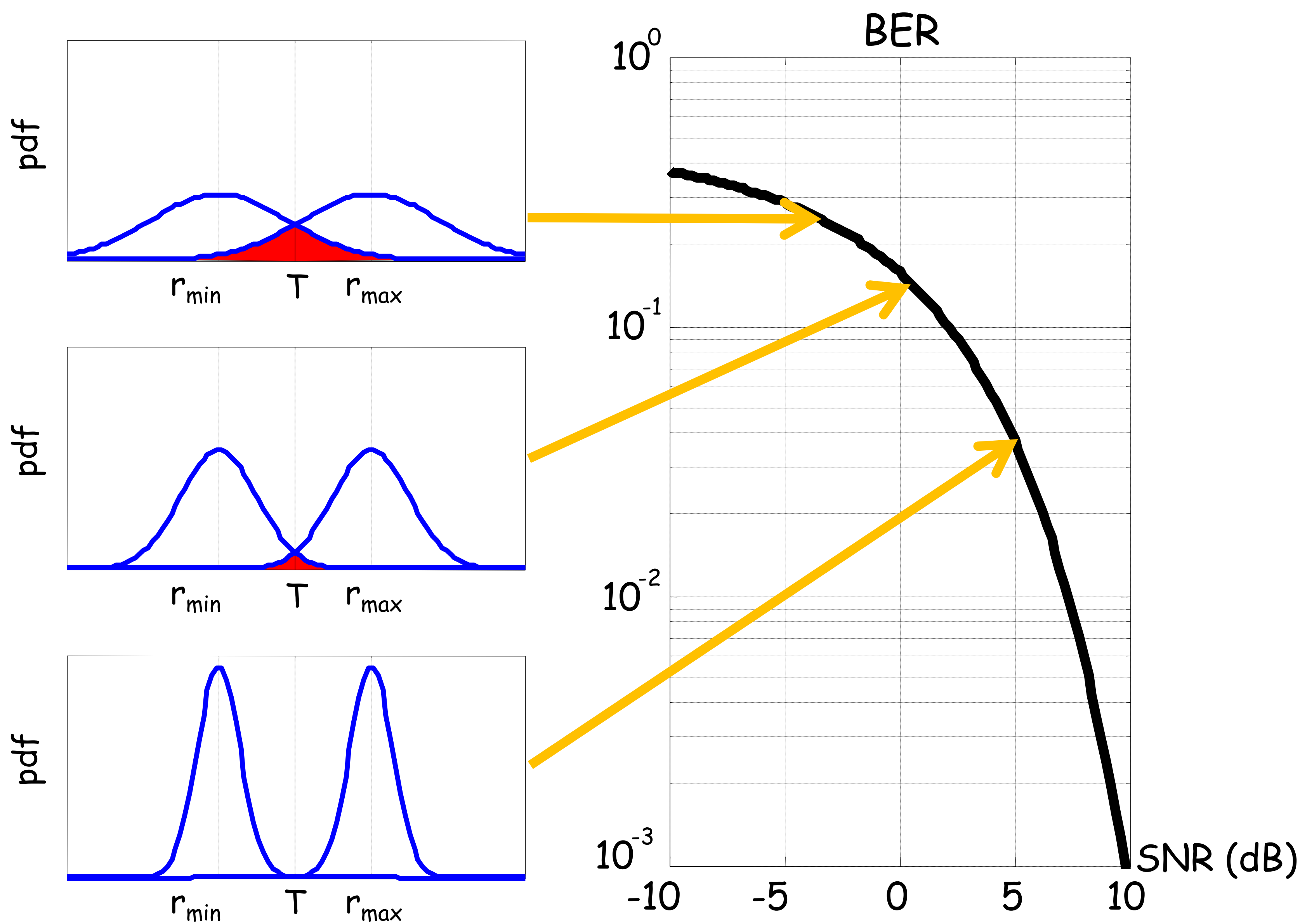
- Received power decreases as the receiver moves away
- The noise remains pretty constant
- Decrease in received signal power leads to decreased SNR
- Once SNR falls below around 10dB, the receiver will stop functioning- for a mobile phone this is around 10^{-14}W



BER under changing Signal Power



BER under changing Noise Power



Summary

- In digital systems noise leads to bit errors
- Noise can be modeled by statistical distributions
- By using our models of noise it is possible to find exact expressions for the Bit Error Rate (BER) as
  - The signal power changes
  - The noise power changes
  - The threshold changes
- The key parameter determining BER is the Signal to Noise Ratio (SNR)
- BER decreases as SNR increases