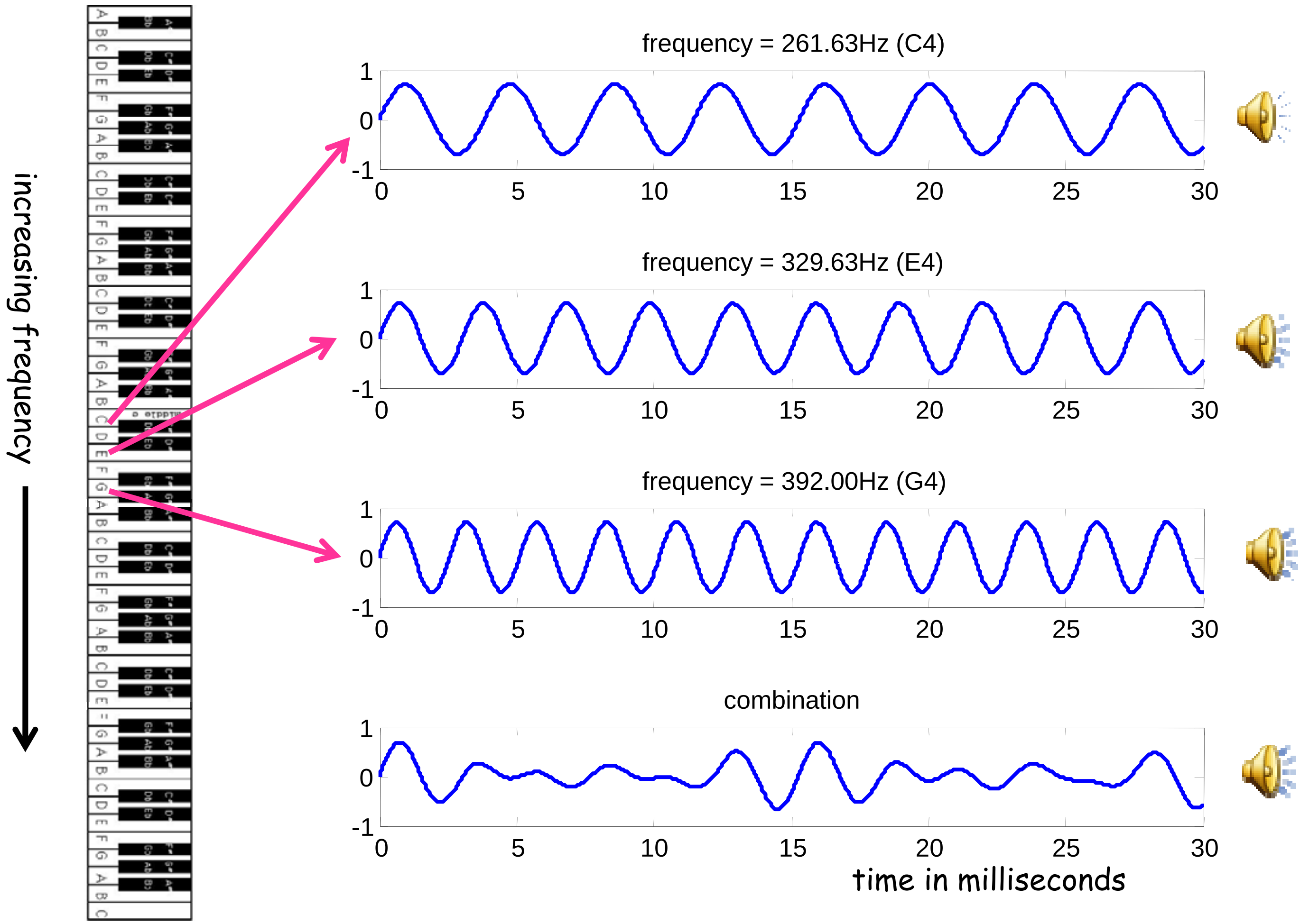
ELEC1200: A System View of Communications: from Signals to Packets
Lecture 12

- **Motivation**
  - Music as a combination of sounds at different frequencies
- **Cosine waves**
  - Continuous time
  - Discrete time
- **Decomposing signals into a sum of cosines**
  - Amplitude Spectrum

Motivation: Music

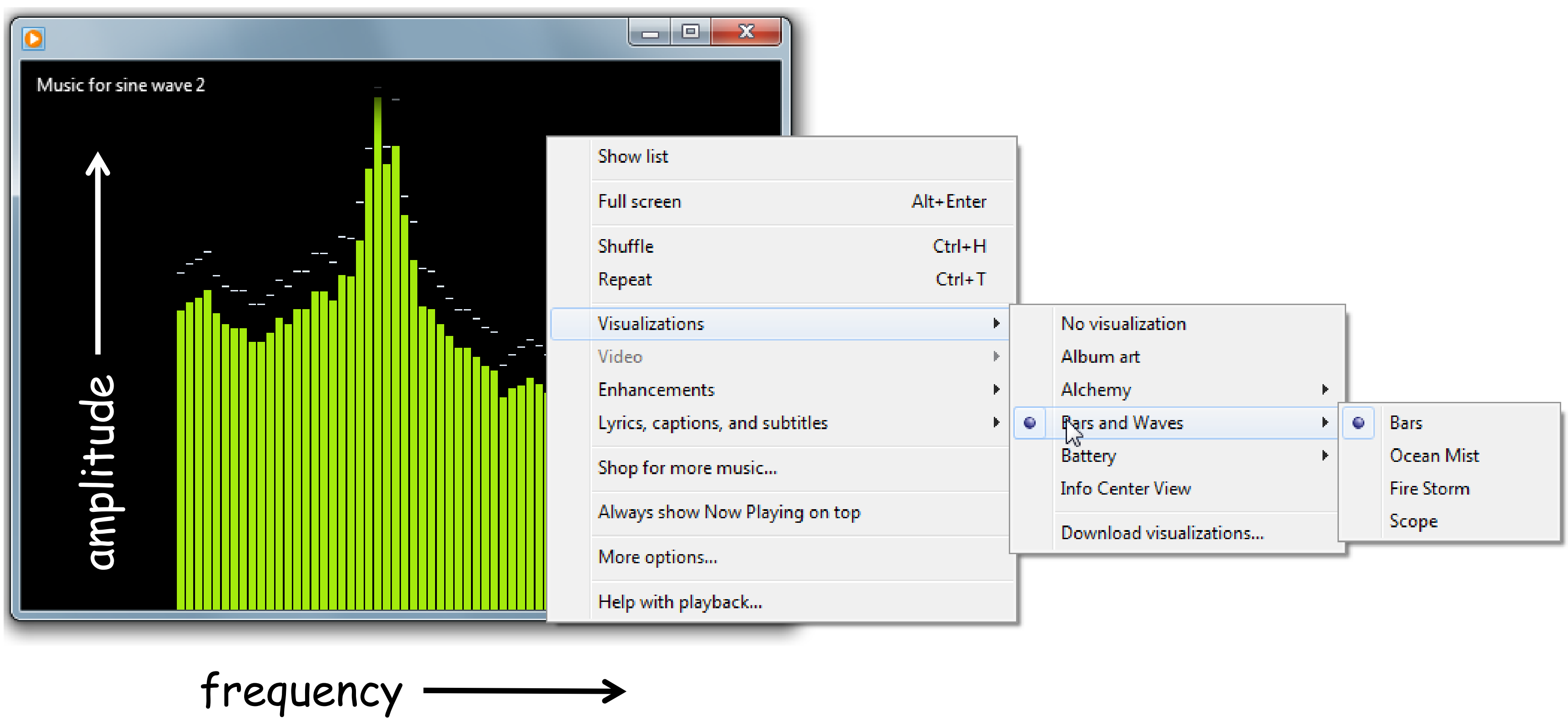
- Music is a combination of sounds with different frequencies:



- In this lecture, we see that this is true of any waveform

Visualizing the frequency content of music

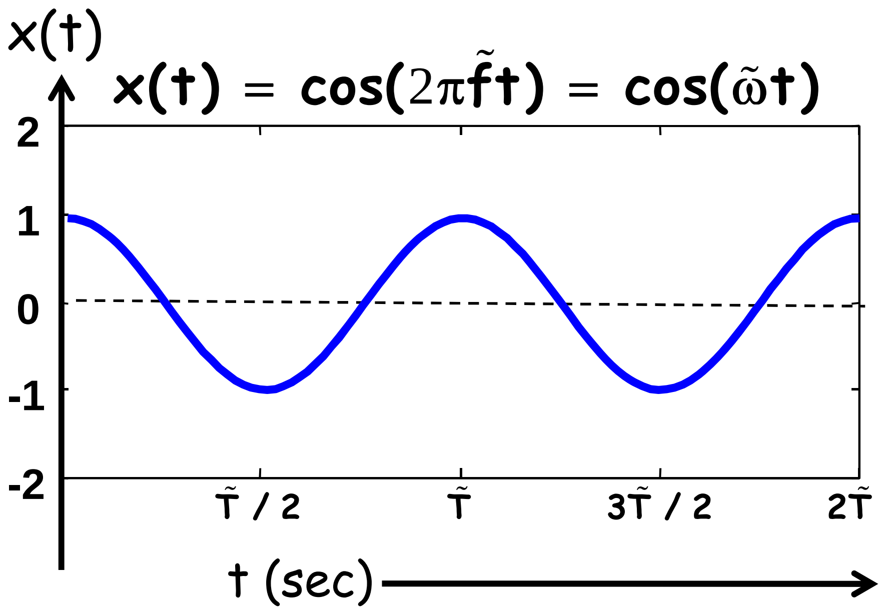
- Windows Media Player
- Right click->Visualizations->Bars and Waves->Bars



Sinusoidal Waves

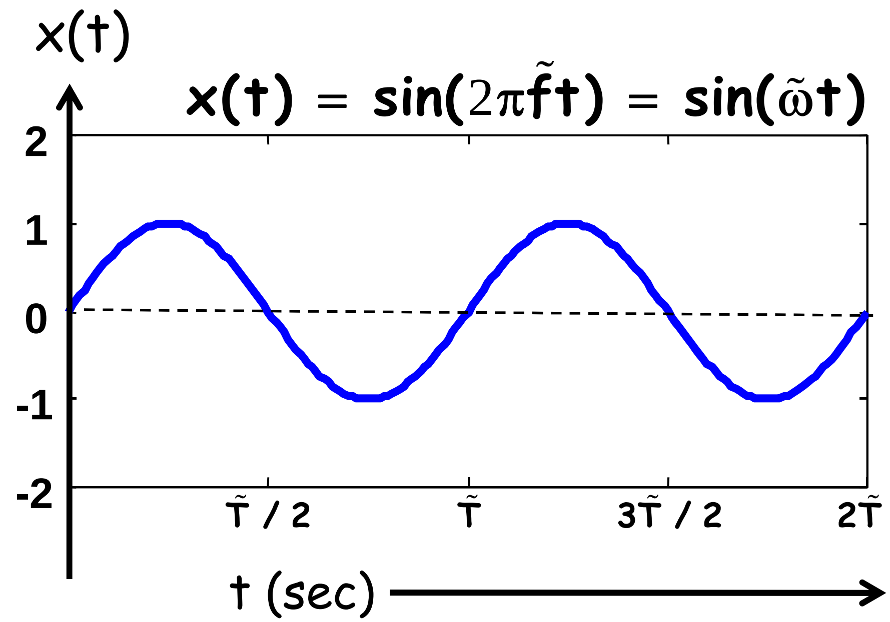
Cosine Wave

$$x(t) = \cos(2\pi\tilde{f}t) = \cos(\tilde{\omega}t)$$



Sine Wave

$$x(t) = \sin(2\pi\tilde{f}t) = \sin(\tilde{\omega}t)$$



Parameters:

$\tilde{T}$ = period (in seconds)

$\tilde{f}$ = frequency in hertz (cycles per second)

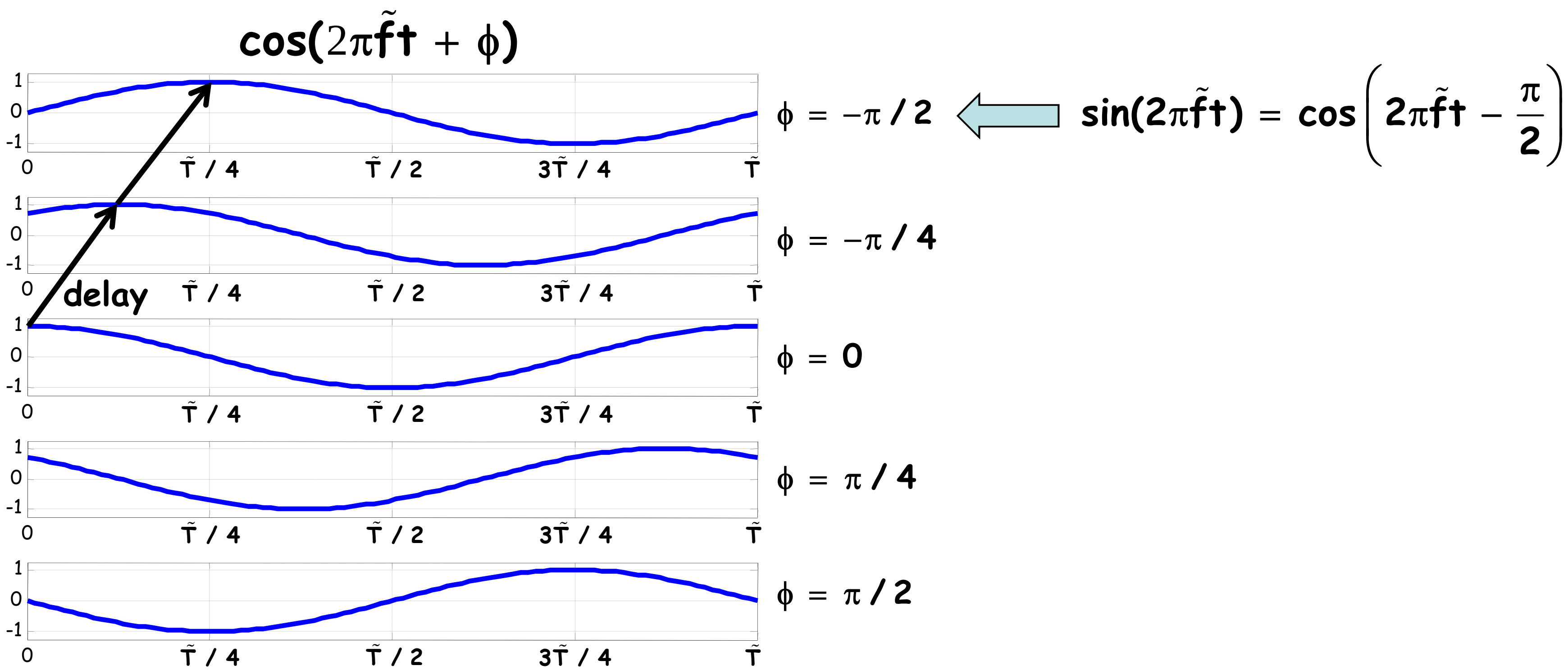
$\tilde{\omega}$ = frequency in radians per second

$$\tilde{f} = \tilde{T}^{-1}$$

$$\tilde{\omega} = 2\pi\tilde{f}$$

We only need cosines

- A sine wave is just a cosine with a phase shift of $-\frac{\pi}{2}$

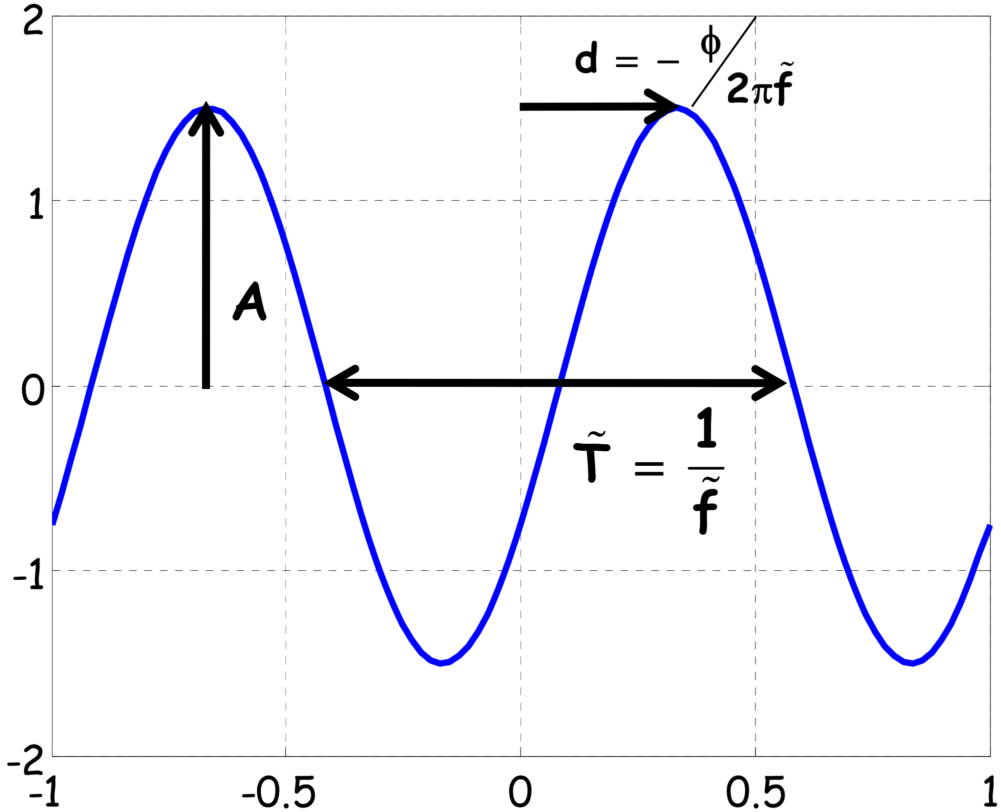


- A negative phase shift introduces a delay of $d = \frac{-\phi}{2\pi\tilde{f}} = \frac{-\phi\tilde{T}}{2\pi}$

$$\cos(2\pi\tilde{f}t + \phi) = \cos\left(2\pi\tilde{f}\left(t - \frac{-\phi}{2\pi\tilde{f}}\right)\right) = \cos\left(2\pi\tilde{f}\left(t - d\right)\right)$$

General Form

- Any sinusiodal signal can be expressed as



$$x(t) = A\cos(2\pi\tilde{f}t + \phi)$$

A = amplitude

$\tilde{f}$ = frequency

ϕ = phase

Discrete Time Cosines

- Consider a discrete time cosine waveform with N samples:

$$\cos(2\pi fn + \phi) \text{ for } n = 0, 1, \ldots (N-1)$$

- In the following, we assume N is even.
- If we have only N samples, it turns out that we only need to consider N/2+1 frequencies:

$$f_k = \frac{k}{N} \text{ for } k \in \left\{0, 1, \ldots \tfrac{N}{2}\right\} \longleftarrow$$

f_k is called a normalized frequency
It has units of cycles/sample

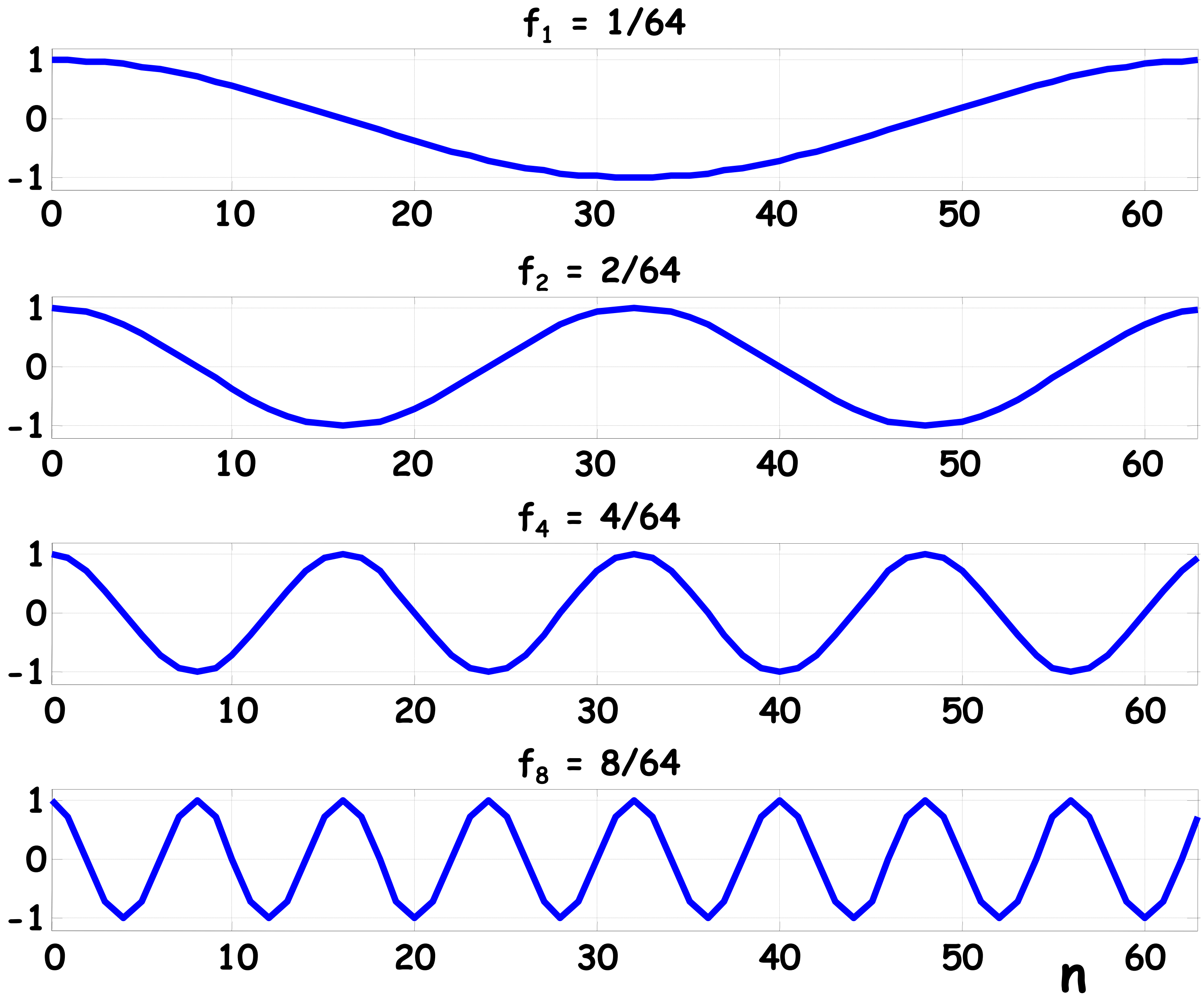
- For all N, it is always true that $0 \leq f_k \leq 0.5$

- k indicates how many times the cosine repeats in N samples

$$\cos(2\pi f_k n) = \cos(2\pi k \frac{n}{N})$$

- The larger k, the higher the frequency

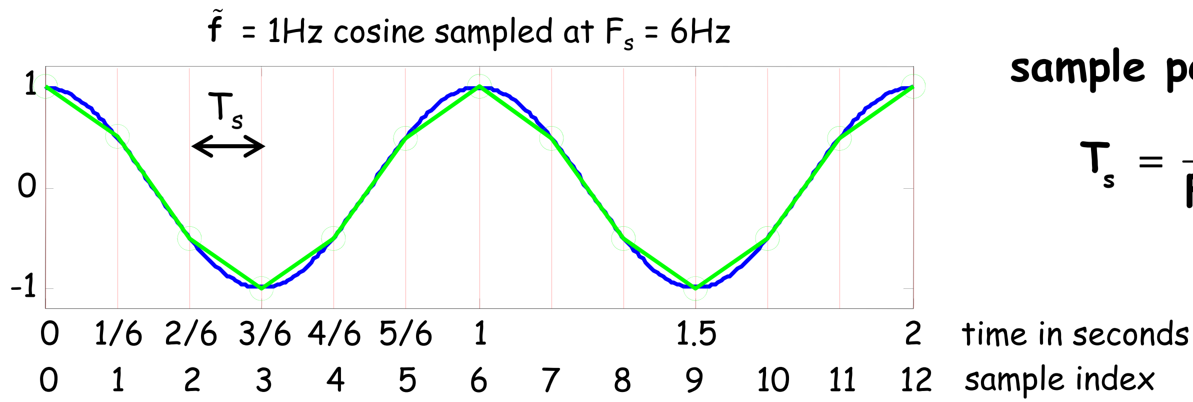
Discrete time cosines with varying f_k



Normalized vs continuous frequency

- Given a continuous cosine of frequency $\tilde{f}$ and sampling at frequency F_s, we get a set of samples $x(n)$:

$$
\begin{aligned}
x(n) &= \cos(2\pi\tilde{f}\,\frac{n}{F_s}) \\
&= \cos(2\pi\,\frac{\tilde{f}}{F_s}\,n) \\
&= \cos(2\pi f n)
\end{aligned}
$$



sample period

$$T_s = \frac{1}{F_s}$$

- Units of f are cycles per sample: $\dfrac{\tilde{f}\ (\text{Hz})}{F_s\ (\text{Hz})} = \dfrac{\text{cycles}/\text{sec}}{\text{samples}/\text{sec}} = \dfrac{\text{cycles}}{\text{samples}}$

- Conversion:

frequency in Hz

$$f = \frac{\tilde{f}}{F_s} \qquad\qquad \tilde{f} = f \cdot F_s$$

normalized frequency

Sinusoidal Representation of Sampled Data

- Any sampled data waveform x(n) with N samples:

$$x(n) \text{ for } n = 0, 1, \dots (N-1)$$

can be expressed as the the sum of a N/2+1 cosine waves with frequencies

$$f_k = \frac{k}{N} \text{ for } k \in \left\{0, 1, \dots \frac{N}{2}\right\}$$

using the equation

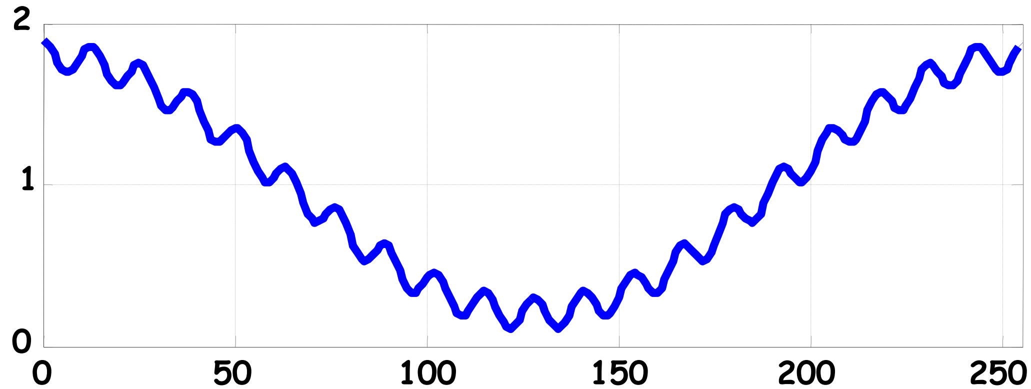
each term in the summation is called a <u>frequency component</u>

$$x(n) = A_0 + \sum_{k=1}^{N/2} A_k \cos(2\pi f_k n + \phi_k)$$

- This equation is called the Fourier series expansion of x(n)
- Different waveforms are obtained by changing the values of A_k and ϕ_k

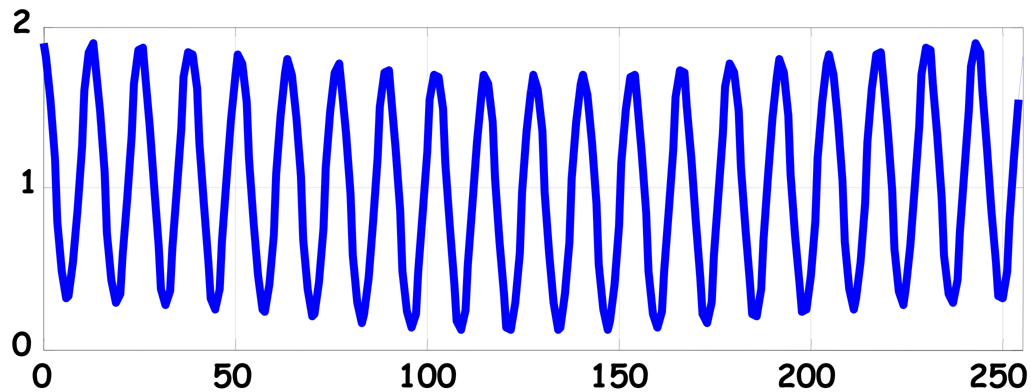
Interpretation of $x(n) = \sum_{k=0}^{N/2} A_k \cos(2\pi f_k n + \phi_k)$

- A_k controls the amplitude of the cosine with frequency f_k.
- ϕ_k controls the phase of the cosine with frequency f_k.



$A_0 = 1$
$A_1 = 0.8$
$A_{20} = 0.1$

large low frequency component, small high frequency component



$A_0 = 1$
$A_1 = 0.1$
$A_{20} = 0.8$

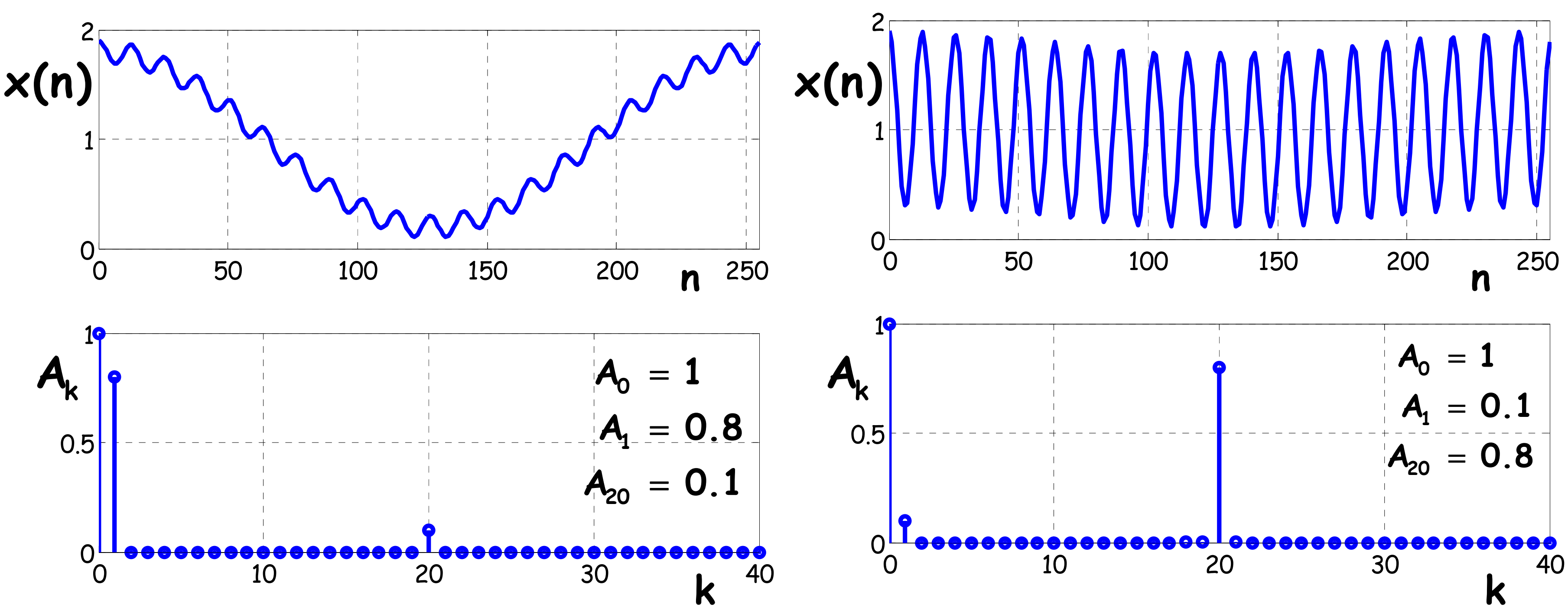
small low frequency component, large, high frequency component

Amplitude, Phase and Frequency

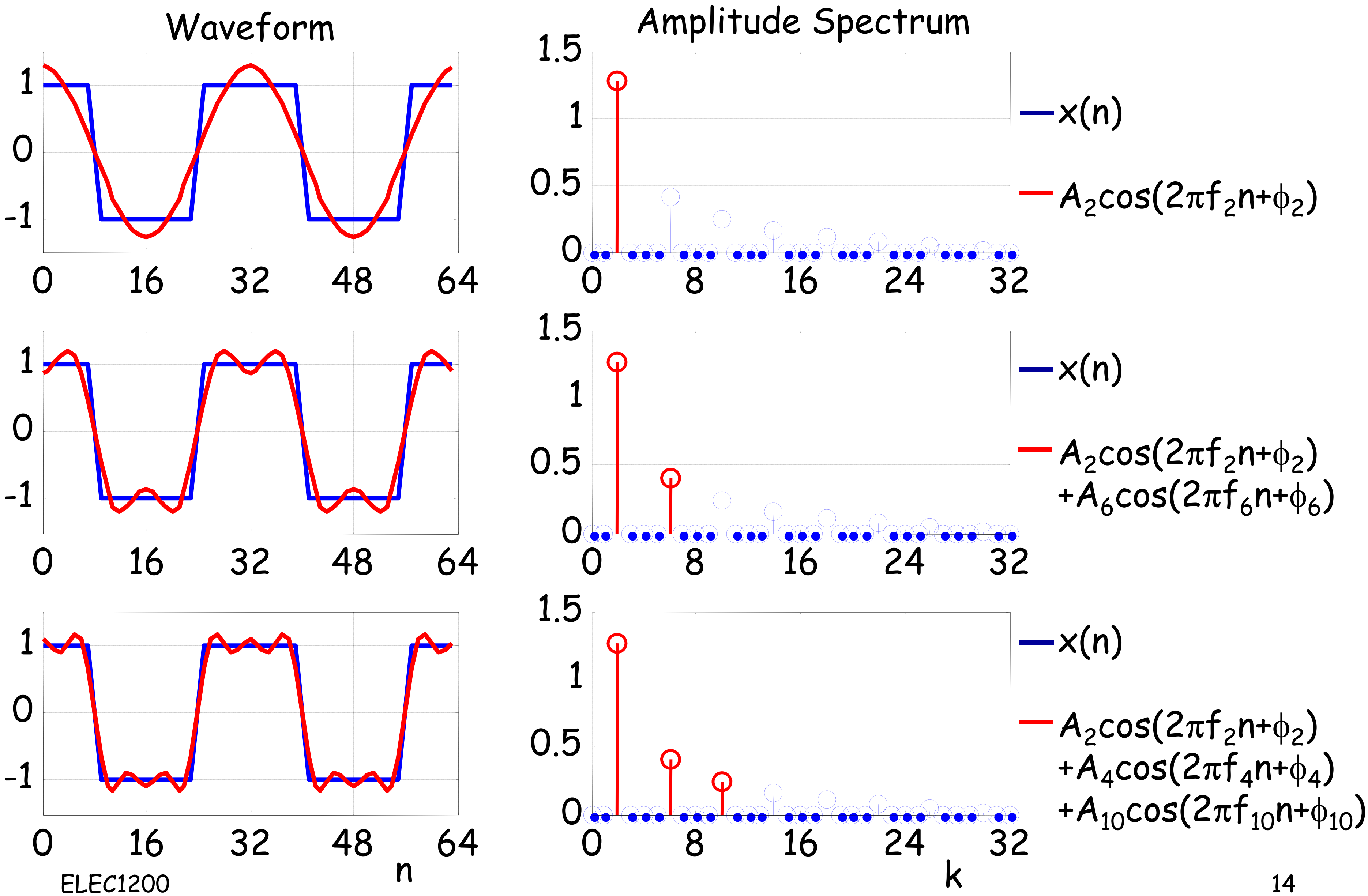
- **Amplitude A_k**
  - tells us how much of each frequency there is.
  - A_0 controls the average level of the signal and is sometimes called the DC component.
- **Phase ϕ_k**
  - Not so important, just shifts each cosine left or right.
- **Frequency f_k**
  - tells us about how quickly the signal changes
  - Low frequencies (small k) indicate slow changes.
  - High frequencies (large k) indicate fast changes.
- **We are most interested in the amplitude and how it changes with k.**
  - The frequency components with the largest amplitude are the "most important."

Amplitude Spectrum

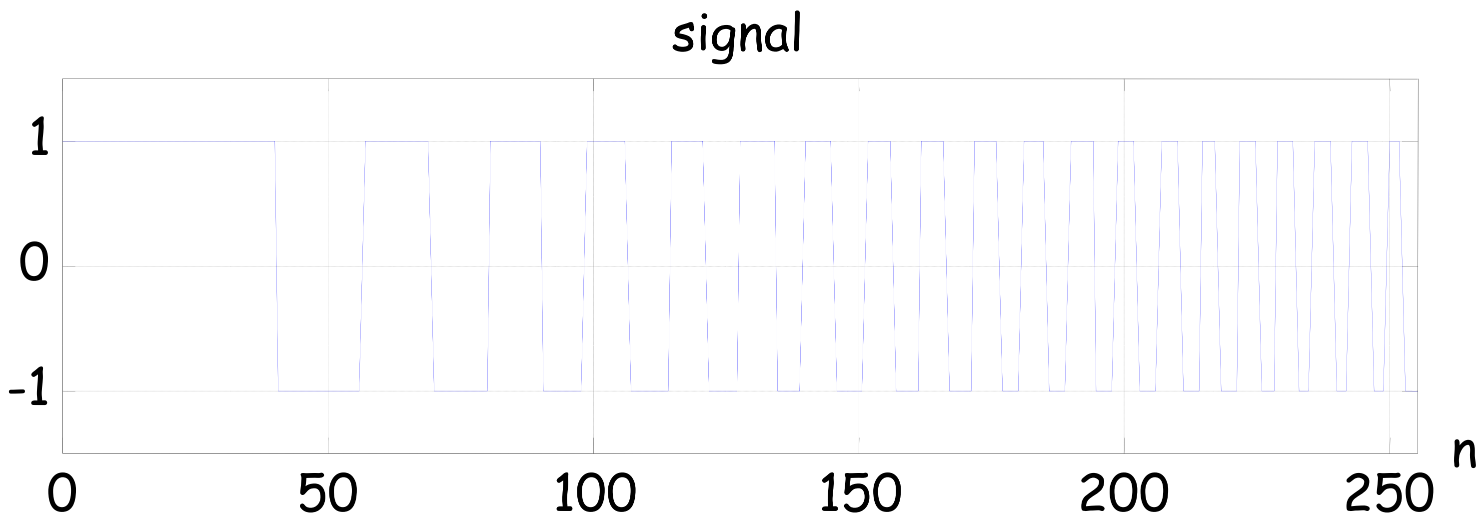
- A plot of A_k versus k
- Gives a graphical representation of which frequencies are most important

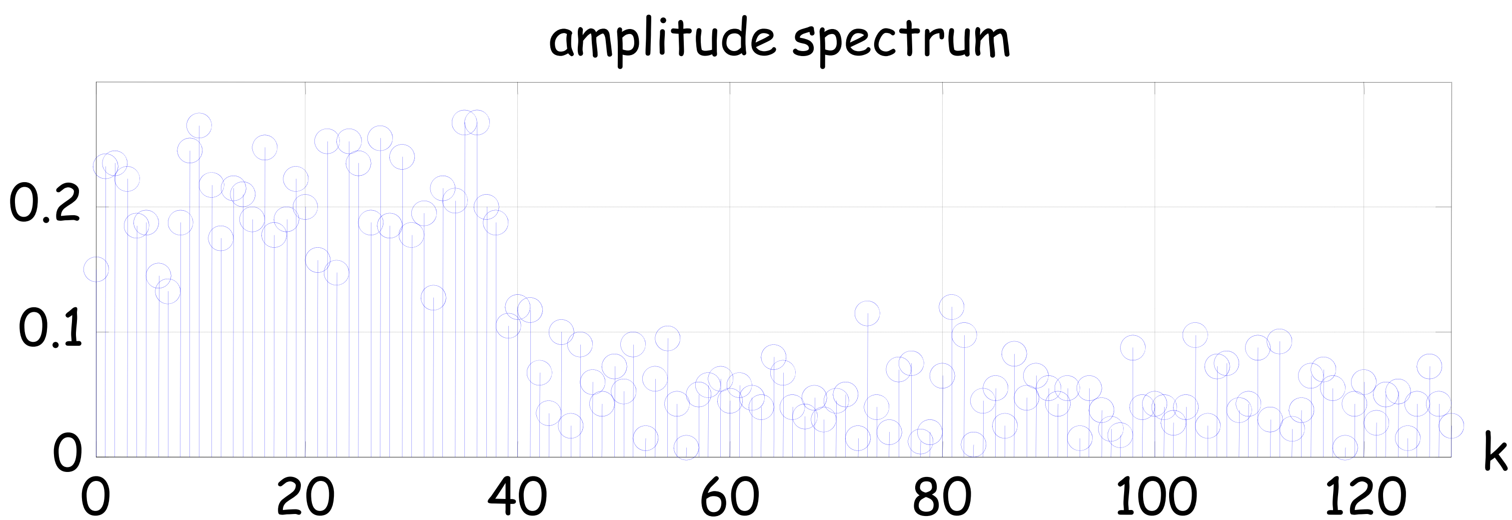


Example: Square Wave



More Complex Example





Transforms

- The Fourier Series is only one of a set of representations (transforms) you will study later:
  - Fourier Transform
  - Laplace Transform
  - Z Transform
- Transforms are merely a different way of expressing the same data. No information is lost or gained when taking a transform.
  - Loose analogy: transform ~ translation
- We use transforms because
  - It gives us a different way of viewing or understanding the signal.
  - Some operations on signals are easier to understand/analyze after taking the transform.

Summary

- Any signal can be expressed as the sum of cosines with different frequencies.
- Cosines in discrete time use normalized frequency, which has units of cycles/sample
- The amplitude spectrum
  - Is the weightings (sizes) of the different frequency components, A_k, plotted as a function of k
  - Tells us important information about what the signal looks like.
  - Frequency components with the largest amplitudes are the "most important"
- Knowledge of both the amplitude and phase spectrum is equivalent to knowledge of the signal.

Appendix: Discrete Fourier Transform (DFT)

- **The DFT takes a waveform x(n) and returns a set of N complex valued coefficients, called <u>Fourier coefficients</u>.**
  - The Fourier coefficients are denoted by X_k for k=0,1…(N-1)
  - The formula (not important for this course) is

$$X_k = \sum_{n=0}^{N-1} x(n)\underbrace{e^{-j\frac{2\pi kn}{N}}} \quad \text{for } k = 0, 1, \ldots, (N-1)$$

This is the complex exponential
For more details wait until you take ELEC2100

  - For this course, we use MATLAB's "fft" function to compute the Fourier coefficients.

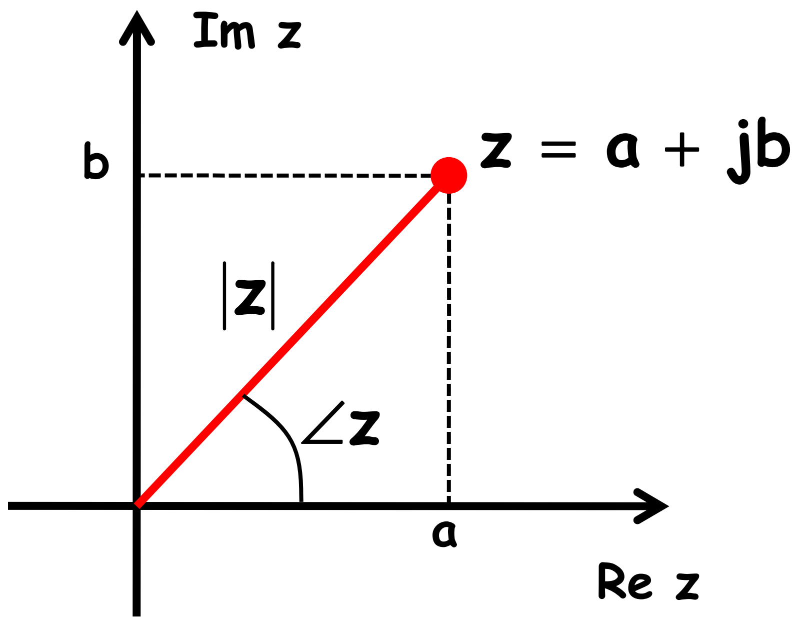
- **From the first N/2+1 Fourier coefficients, we can compute A_k and ϕ_k for k=0,1,…(N/2)**

$$A_k = \begin{cases} \frac{1}{N} X_k & k = 0 \text{ or } N/2 \\ \frac{2}{N} \left| X_k \right| & \text{otherwise} \end{cases} \qquad \phi_k = \angle X_k$$

Appendix: complex numbers

- Define $j = \sqrt{-1}$
- A complex number is given by z = a+jb, where
  - a is the "real" part = Re{z}
  - b is the "imaginary" part = Im{z}
- Since a complex number has two parts, we can represent it as a point on a 2D plane (the complex plane)
- By analogy with polar coordinates, we can define the magnitude and phase.

$$|z| = \sqrt{a^2 + b^2}$$
$$\angle z = \arctan(b / a)$$
$$a = |z| \cos(\angle z)$$
$$b = |z| \sin(\angle z)$$



Appendix: The DFT in MATLAB

- The MATLAB function "`fft`" implements the "Fast Fourier Transform," a fast way to compute the DFT.

  ```
  >> Xdft = fft(x);
  ```

  returns an N dimensional vector containing the DFT coefficients.
- We need to be a bit careful with MATLAB's indexing
  - The first index of $x(n)$ in our notation is $n=0$, but the first index of a vector in MATLAB starts with `1`.
  - Thus, if we represent a waveform $x(n)$ for $n=0,..(N-1)$ in a MATLAB vector `x`. Then,

    `x(1)`= $x(0)$, `x(2)`= $x(1)$,…, `x(N)`= $x(N-1)$

    MATLAB signal
  - Similarly,

    `Xdft(1)` = X_0