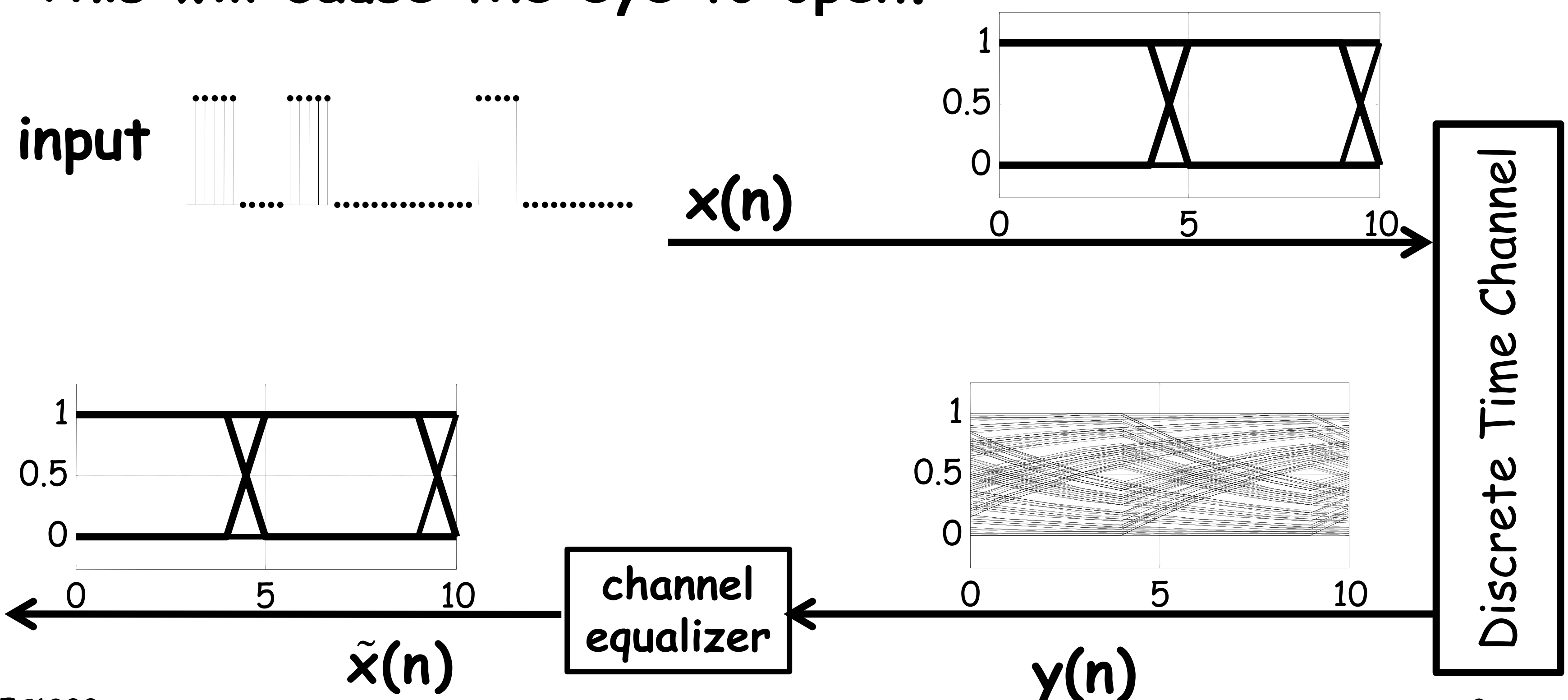
ELEC1200: A System View of Communications: from Signals to Packets
Lecture 6

- **Motivation**
  - **Developing an equalizer to "open" the "eye"**
- **Modeling the IR channel using a recursive equation**

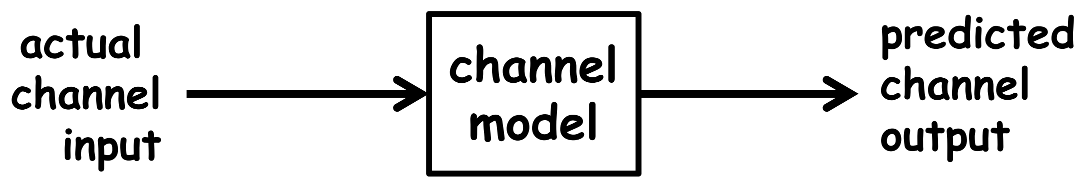
Motivation: Equalization

- The channel introduces intersymbol interference, which causes the eye to close.
- The goal of a channel equalizer is to “undo” the effect of the channel.
- This will cause the eye to open.

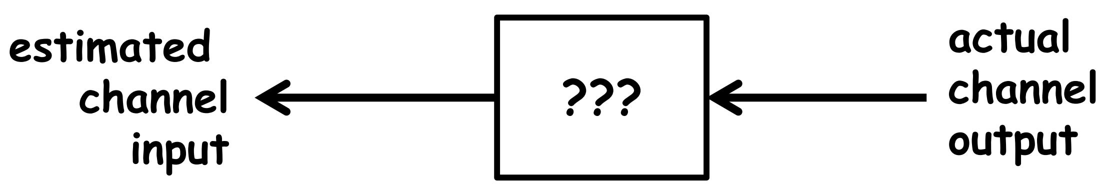


Developing the Equalizer

- How can we figure out what to put into the channel equalizer?
  - We need to describe the effect of the channel using a model that enables us to predict the channel output for any input.



  - Using this model, we can hopefully *guess* (estimate) what the input to the channel was from knowledge of the output

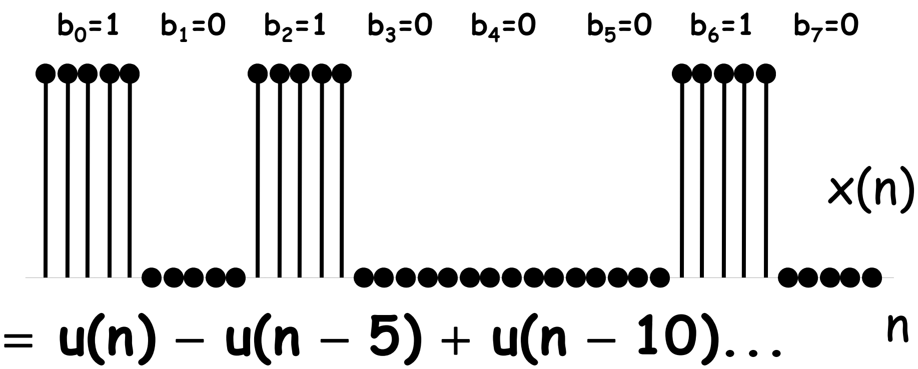


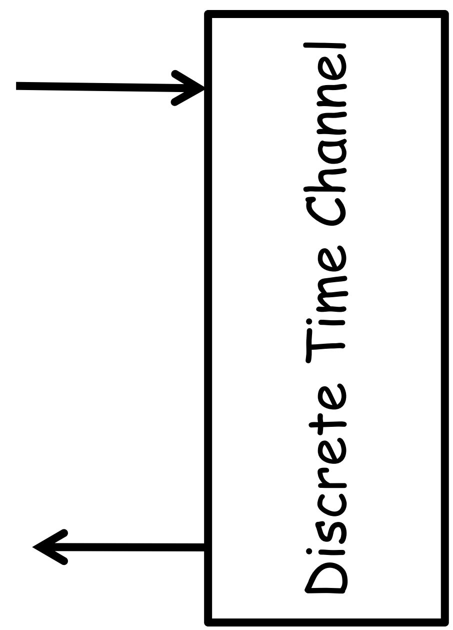
Models

- A model is a way of describing something, e.g. a channel.
- Models are used
  - To capture (express) our understanding
  - To make predictions
  - To make decisions (take action!)
- In engineering and science, models often take the form of mathematical equations.
- One system may be described by many models that are expressed differently, but are equivalent (they make the same prediction for the output)
  - The more models we have, the better we understand the system.
  - Different models have different uses.

Model 1

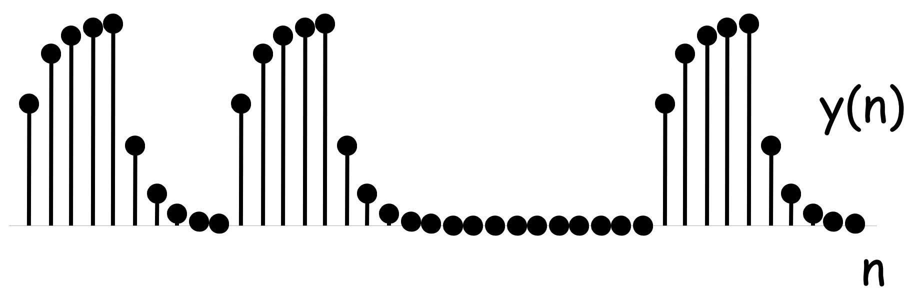
- The communication channel can be modeled by
  - Assuming it is linear and time invariant (LTI)
  - Assuming it has known step response.





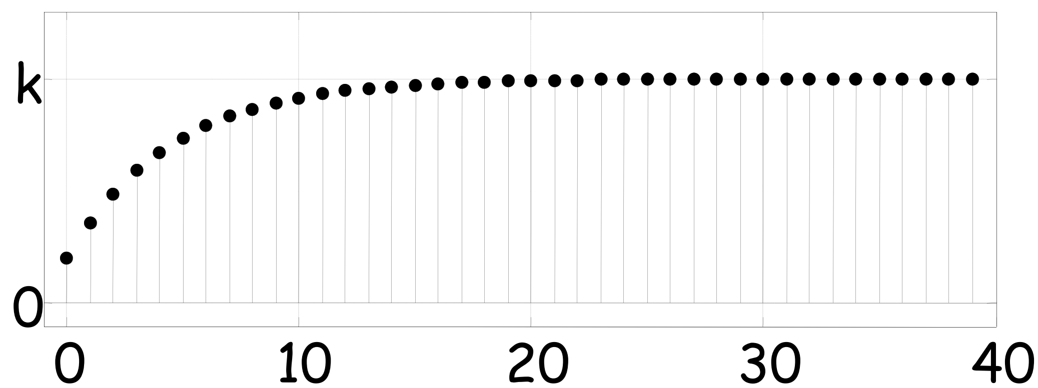
$$x(n) = u(n) - u(n-5) + u(n-10)\ldots$$

$$y(n) = s(n) - s(n-5) + s(n-10)\ldots$$



Model 1 (cont.)

- **The step response of the IR channel can be described by an exponential response**



$$s(n) = k(1-a^{n+1})u(n)$$

$$0 < a < 1$$

- **This model is useful in predicting the output for a given input.**
- **However, it is not so easy to use this model to go the other way (predict the output from the input).**
- **Thus, we seek another equivalent model.**

Recursive model

- A sequence is a set of numbers that are indexed (ordered) by an integer n, e.g.
  - $x(1), x(2), x(3), \ldots, x(n)$
- A sequence is often specified as a function of n, e.g. $x(n) = n$
- A recursive model for a sequence has two parts
  - a formula that defines the nth sample in terms of the past samples, e.g.

$$x(n) = f(x(n-1))$$

  - an initial (starting) condition, e.g.

$$x(0) = 0$$

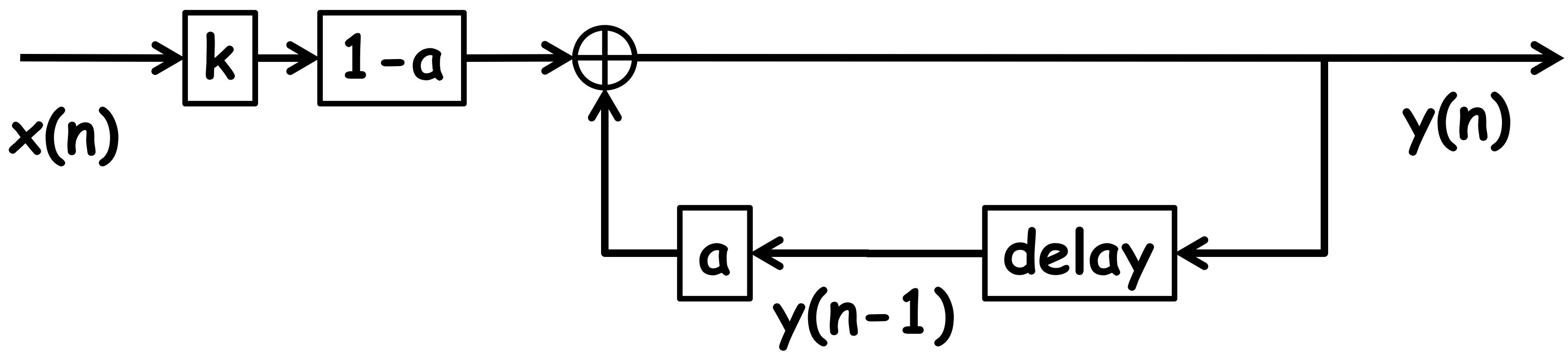
Examples

- **Can you think of recursive models for the following sequences:**
  - **$x(n) = c$ (a constant)**
  - **$x(n) = n$ (a linear ramp)**
  - **$x(n) = 0.2n$**
  - **$x(n)$ = an alternating bit stream (0 1 0 1 0 1…)**

Recursive model of IR channel

- It turns out that the response of the IR channel y(n) to an input x(n) can be described by a recursive formula:

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$



- The parameter a lies between 0 and 1.
- The parameter k scales the input.
- This is also known as a "feedback" system, since the output feeds back as an input to the system to determine the next output.

Effect of Parameter a

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

- The parameter a determines the “memory” in the channel
- a = 0:
  - no memory of the past
  - $y(n) = k \cdot x(n)$
  - the channel output is just the input multiplied by k.
- a = 1
  - infinite memory of the past
  - $y(n) = y(n-1)$
  - Channel output is constant, ignores the channel input.

Equivalent models

- The following two models of the channel are equivalent in the sense that they predict the same output for a *given* input:
  - Model 1: The channel is an LTI system with step response
    $$s(n) = k(1-a^{n+1})u(n)$$
    where $0 < a < 1$
  - Model 2: The channel has input $x(n)$ and output $y(n)$ determined by
    $$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$
    where $0 < a < 1$.

- We can establish equivalence by showing that
  - Model 2 is LTI (linear and time invariant)
  - Model 2 has a response to a unit step given by $s(n)$

Fact 1: Model 2 is LTI

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

- **It is linear**
  - The output is a linear function of the input and past outputs.
  - It satisfies the two properties of
    - Homogeneity
    - Additivity
- **It is time invariant**
  - The parameters k and a do not change with n.

Same Step Response (example)

Model 1:

$s(n) = k(1-a^{n+1})u(n)$

Model 2:

$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$

Assume: $x(n) = u(n)$, $y(-1) = 0$, $a = \frac{1}{2}$, $k = 1$

n	Model 1 $s(n)=1-(\frac{1}{2})^{n+1}$	Model 2 $y(n) = \frac{1}{2}y(n-1)+\frac{1}{2}$
0		
1		
2		
3		
4		
5		

Summary

- We have seen that the channel can be described (modelled) in two equivalent ways
  - Model 1:
    - Channel is linear and time invariant
    - Channel has step response $s(n) = k(1-a^{n+1})u(n)$
  - Model 2:
    - If $x(n)$ is the channel input and $y(n)$ is the output,

$$y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot x(n)$$

- In the next lecture, we will see how we can use Model 2 to "undo" (equalize) the effect of the channel, and "open" the eye.

Rigorous Proof of Equivalence (Optional!)

We use recursion to show that the response of model 2 to the unit step is the step response of model 1.

Let $x(n) = u(n)$ in model 2, i.e. $y(n) = a \cdot y(n-1) + (1-a) \cdot k \cdot u(n)$

1. Since $u(n) = 0$ for all $n<0$,

$$y(n) = a \cdot y(n-1) \text{ for all } n < 0$$

   This is satisfied if $y(n) = 0$ for all $n<0$.

2. At $n = 0$,

$$\begin{aligned} y(0) &= a \cdot y(-1) + (1-a) \cdot k \cdot u(0) \\ &= k(1-a) \end{aligned}$$

3. At $n = 1$,

$$\begin{aligned} y(1) &= a \cdot y(0) + (1-a) \cdot k \cdot u(1) \\ &= k(a-a^2) + k(1-a) = k(1-a^2) = k(1-a^{n+1}) \end{aligned}$$

4. Suppose that $s(n-1) = k(1-a^n)$,

$$\text{then, } s(n) = k(a - a^{n+1}) + k(1-a) = k(1-a^{n+1})$$

Thus, by recursion induction, $s(n) = k(1-a^{n+1})u(n)$ for all n.